The importance of being uncomplicated

Abstract:

By Ramsey's theorem, any system of \( n \) segments in the plane has roughly \( \log n \) members that are either pairwise disjoint or pairwise intersecting. Analogously, any set of \( n \) points \( p(1),...,p(n) \) in the plane has a subset of roughly \( \log \log n \) elements with the property that the orientation of the triangle \( p(i)p(j)p(k) \) is the same for all triples from this subset with \( i<j<k \). The elements of such a subset form the vertex set of a convex polygon. However, in both cases we know that there exist much larger "homogeneous" subsystems satisfying the above conditions. What is behind this favorable behavior? One of the common features of the above problems is that the underlying graphs and triple-systems can be defined by a small number of polynomial equations and inequalities in terms of the coordinates of the segments and points. We discuss some structural properties of "semi-algebraically" defined graphs and hypergraphs, including Szemerédi-type partition theorems.