TAME DISCRETE SUBSETS IN COMPLEX ALGEBRAIC GROUPS

JÖRG WINKELMANN

For discrete subsets in $\mathbf{C}^n$ the notion of being "tame" was defined by Rosay and Rudin. A discrete subset $D \subset \mathbf{C}^n$ is called "tame" if and only if there exists an automorphism ϕ of $\mathbf{C}^n$ such that $\phi(D) = \mathbb{N} \times \{0\}^{n-1}$.

We are interested in similar notions for complex manifolds other than $\mathbf{C}^n$.

Therefore we propose a new definition, show that it is equivalent to that of Rosay and Rudin if the ambient manifold is $\mathbf{C}^n$ and deduce some standard properties.

To obtain good results, we need some knowledge on the automorphism group of the respective complex manifold. For this reason we get our best results in the case where the manifold is biholomorphic to a complex Lie group.

Definition. *Let X be a complex manifold. An infinite discrete subset D is called (weakly)* tame *if for every exhaustion function $\rho : X \to \mathbb{R}^+$ and every map $\zeta : D \to \mathbb{R}^+$ there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.*

Andrist and Ugolini have proposed a different notion, namely the following:

Definition. *Let X be a complex manifold. An infinite discrete subset D is called (strongly)* tame *if for every injective map $f : D \to D$ there exists an automorphism ϕ of X such that $\phi(x) = f(x)$ for all $x \in D$.*

It is easily verified that "strongly tame" implies "weakly tame". For $X \simeq \mathbf{C}^n$ both tameness notions coincide with each other and with the tameness notion of Rosay and Rudin.

In the sequel, unless explicitly stated otherwise, tame always means weakly tame.

For tame discrete sets in $\mathbf{C}^n$ in the sense of Rosay and Rudin, the following facts are well-known:

(1) Any two tame sets are equivalent.

Key words and phrases. tame discrete set, semisimple complex Lie group.

(2) Every discrete subgroup of $(\mathbf{C}^n, +)$ is tame as a discrete set.
(3) Every discrete subset of $\mathbf{C}^n$ is the union of two tame ones.
(4) There exist non-tame subsets in $\mathbf{C}^n$.
(5) Every injective self-map of a tame discrete subset of $\mathbf{C}^n$ extends to a biholomorphic self-map of $\mathbf{C}^n$.
(6) If v_k is a sequence in $\mathbf{C}^n$ with $\sum_{k=1}^{\infty} \frac{1}{||v_k||^{2n-1}} < \infty$, then $\{v_k : k \in \mathbb{N}\}$ is a tame discrete subset.

We prove the following.

Theorem. *Let G be a complex linear algebraic group whose character group is trivial, i.e., such that there is no non-constant morphism of algebraic groups from G to the multiplicative group $\mathbb{C}^*$.*

Let D be a discrete subset.

Then the following conditions are equivalent:

(1) *D is (weakly) tame.*
(2) *D is strongly tame.*
(3) *There exists a biholomorphic self-map ϕ of G and a unipotent subgroup $U \subsetneq G$ such that $\phi(D) \subset U$.*

Furthermore:

(1) *Any two tame discrete sets are equivalent.*
(2) *Every discrete set may be realized as the union of two tame discrete sets.*
(3) *If D is tame, then $G \setminus D$ is an Oka manifold.*
(4) *$SL_n(\mathbb{Z})$ is a tame discrete subset of $SL_n(\mathbb{C})$.*

Jörg Winkelmann, Lehrstuhl Analysis II, Mathematisches Institut, Ruhr-Universität Bochum, 44780 Bochum, Germany, e-mail:joerg.winkelmann@rub.de