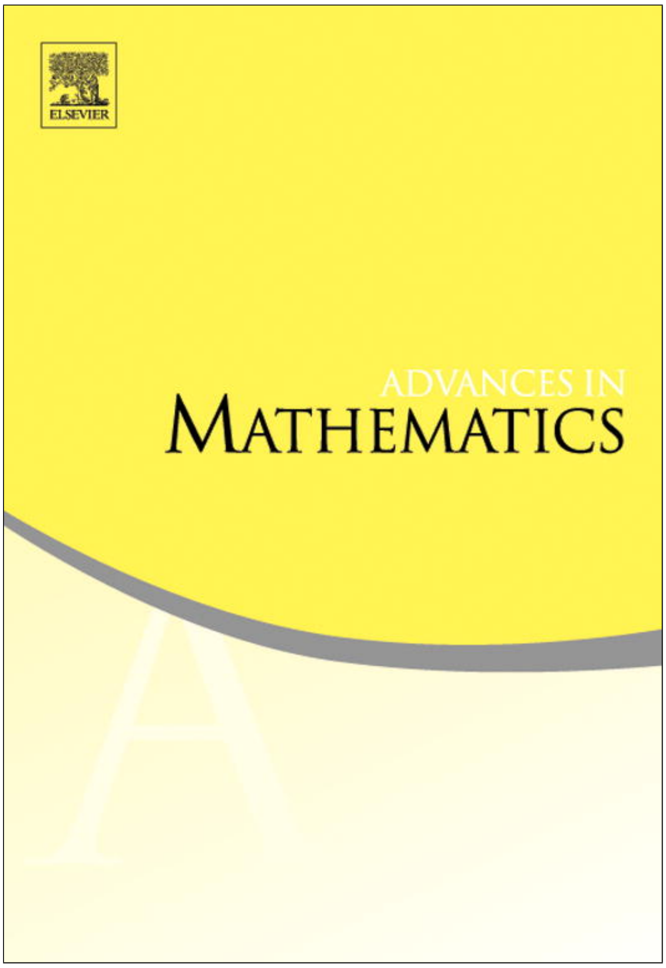
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Projection and slicing theorems in Heisenberg groups

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Abstract

We study the behavior of the Hausdorff dimension of sets in the Heisenberg group $\mathbb{H}^n$, $n \in \mathbb{N}$, with a sub-Riemannian metric under projections onto horizontal and vertical subgroups, and under slicing by translates of vertical subgroups. We formulate almost sure statements in terms of a natural measure on the Grassmannian of isotropic subspaces.

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Contents

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1. Introduction

In this paper we discuss how the Hausdorff dimension of sets in higher dimensional Heisenberg groups equipped with a non-Euclidean metric of sub-Riemannian type behaves under projections onto horizontal and complementary vertical subgroups, and under slicing by translates of vertical subgroups. Our results contribute to the ongoing study of the internal metric and measure-theoretic structure of sub-Riemannian spaces. Such a program was originally formulated in Gromov's groundbreaking treatise [10].

Our studies are motivated by the intention to find Heisenberg counterparts of classical almost sure statements in geometric measure theory. In the Euclidean space $\mathbb{R}^n$ such results are formulated in terms of a natural measure $\gamma_{n,m}$ on the *Grassmannian* $G(n,m)$ of all m-dimensional linear subspaces of $\mathbb{R}^n$ for integers $0 < m < n$.

Denote by $\pi_V : \mathbb{R}^n \to V$ the Euclidean orthogonal projection from $\mathbb{R}^n$ onto a subspace $V \in G(n,m)$ and let $A \subset \mathbb{R}^n$ be a Borel set. It was proved by Marstrand [14] (for the case $n = 2$) and by Mattila [15] that

$$\dim_E \pi_V(A) = \min\{\dim_E A, m\} \quad \text{for } \gamma_{n,m} \text{ almost every } V. \tag{1.1}$$

Moreover, the Hausdorff m-dimensional measure $\mathcal{H}^m_E(\pi_V(A))$ is positive for $\gamma_{n,m}$ almost every $V \in G(n,m)$ if $\dim_E A > m$.

Closely related to the preceding statements are results estimating the Hausdorff dimension of the intersection of A with translates of the orthocomplement $V^\perp$ of V. If $t = \dim_E A > m$, it follows from the projection theorem that there exists an $\mathcal{H}^m$ positive measure set of parameters $u \in V$ such that $A \cap V_u^\perp \neq \emptyset$, where $V_u^\perp = \pi_V^{-1}\{u\} = V^\perp + u$. Yet more can be said about the dimension of these intersections. In [14] and [15] one finds the estimate

$$\mathcal{H}^m_E(\{u \in V : \dim_E(A \cap V_u^\perp) = t - m\}) > 0 \quad \text{for } \gamma_{n,n-m} \text{ a.e. } V^\perp \in G(n, n-m).$$

As mentioned above, our goal in this paper is to establish analogous results in the sub-Riemannian Heisenberg group. Let us recall that the Heisenberg group $\mathbb{H}^n$ is the unique simply connected, connected nilpotent Lie group of step two and dimension $2n+1$ with one-dimensional center. As a manifold, we may identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. Points $p \in \mathbb{H}^n$ are written in coordinates as

$$p = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}.$$

Denoting $z = (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$, the group law is given by

$$p * p' = (z, t) * (z', t') = (z + z', t + t' + 2\omega(z, z'))$$

with the standard symplectic form $\omega(z, z') = \sum_{i=1}^n y_i x_i' - y_i' x_i$.

In this paper, we give almost sure estimates for the Hausdorff dimension of subsets of $\mathbb{H}^n$ with respect to the *Heisenberg metric* (or *Korányi metric*)

$$d_H(p, p') := \|p^{-1} * p'\|_H \quad \text{with } \|p\|_H = (|z|^4 + t^2)^{1/4}, \tag{1.2}$$

where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^{2n}$. Although the metric d_H induces the Euclidean topology, the properties of the metric space $(\mathbb{H}^n, d_H)$ are substantially different from those of the underlying Euclidean space. For instance, the Hausdorff dimension of $(\mathbb{H}^n, d_H)$ is $2n+2$. It will thus be important to specify the metric with which we are working. We will indicate by a subscript H concepts with respect to the Heisenberg metric, and by a subscript E the corresponding concepts with respect to the Euclidean metric. Thus we denote by $\mathcal{H}^s_H$, resp. $\mathcal{H}^s_E$, the Hausdorff measures and by $\dim_H$, resp. $\dim_E$, the respective Hausdorff dimensions. For sets where the metrics d_H and d_E coincide, we will omit the subscript. We take this opportunity to alert the reader that we will denote by $B(x,r)$ the *closed* ball with center x and radius $r > 0$ in a metric space X. In case $X = (\mathbb{H}^n, d_H)$, resp. $X = (\mathbb{R}^n, d_E)$ we will write $B_H(p,r)$, resp. $B_E(p,r)$.

Note that the Heisenberg metric d_H defined above is bi-Lipschitz equivalent with the usual sub-Riemannian (or Carnot–Carathéodory) metric on $\mathbb{H}^n$. Our conclusions are all invariant under bi-Lipschitz change of metric. Because of its simple explicit form, we work exclusively with the Heisenberg metric d_H in this paper.

In order to describe our results we must identify suitable notions of a Grassmannian of subspaces (or subgroups) in the Heisenberg group, as well as projection mappings into such subspaces. We will consider the class of homogeneous subgroups of the Heisenberg group. A *homogeneous subgroup* $\mathbb{G}$ is a closed subgroup of $\mathbb{H}^n$ which is invariant under the intrinsic dilations $\delta_s(z,t) = (sz, s^2 t)$, $s > 0$.

The homogeneous subgroups can be identified with linear subspaces of $\mathbb{R}^{2n+1}$ which are contained in $\mathbb{R}^{2n} \times \{0\}$ (in which case they are called *horizontal*) or which contain the t-axis (then they are called *vertical*). But not all linear subspaces $\mathbb{V}$ contained in $\mathbb{R}^{2n} \times \{0\}$ are homogeneous subgroups, only those which correspond to *isotropic* subspaces V of $\mathbb{R}^{2n}$, that is, subspaces on which the standard symplectic form vanishes identically. The restriction of the Heisenberg metric to a horizontal subgroup coincides with the Euclidean metric.

Let $\mathbb{V} = V \times \{0\}$ be such a horizontal subgroup. Consider $\mathbb{V}^\perp := V^\perp \times \mathbb{R}$, where $V^\perp$ denotes the Euclidean orthocomplement of V in $\mathbb{R}^{2n}$. It is not hard to see that $\mathbb{V}^\perp$ is a homogeneous (normal) subgroup; we call it the vertical subgroup associated to $\mathbb{V}$. The pair $\mathbb{V}$ and $\mathbb{V}^\perp$ induces a semidirect group splitting $\mathbb{H}^n = \mathbb{V}^\perp \rtimes \mathbb{V}$: each $p \in \mathbb{H}^n$ can be written uniquely as

$$p = P_{\mathbb{V}^\perp}(p) * P_{\mathbb{V}}(p),$$

with $P_{\mathbb{V}^\perp}(p) \in \mathbb{V}^\perp$ and $P_{\mathbb{V}}(p) \in \mathbb{V}$. This gives rise to a well-defined *horizontal projection*

$$P_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}, \quad (z,t) \mapsto P_{\mathbb{V}}(z,t) = (\pi_V(z), 0),$$

and a *vertical projection*

$$P_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp, \quad (z,t) \mapsto P_{\mathbb{V}^\perp}(z,t) = (\pi_{V^\perp}(z), t - 2\omega(\pi_{V^\perp}(z), \pi_V(z))).$$

Whereas horizontal projections correspond to linear projections on the underlying Euclidean space, are Lipschitz continuous both with respect to d_H and d_E, and are group homomorphisms of $\mathbb{H}^n$, vertical projections are neither Euclidean projections, nor Lipschitz continuous, nor group homomorphisms (see [17]). It is therefore more difficult to study the behavior of the Hausdorff dimension of sets in $(\mathbb{H}^n, d_H)$ under vertical projections than under horizontal projections.

Counterparts of the Euclidean projection theorems in the *first* Heisenberg group $\mathbb{H}^1$ were obtained in [2]. The major difference to the Euclidean results is the fact that there is no exact formula to compute the almost sure dimension of projections, but rather a range of possible values. Moreover, dimension can actually *increase* under vertical projections, a phenomenon which is obviously impossible in Euclidean spaces where projections are Lipschitz continuous.

Horizontal subgroups in the first Heisenberg group can be identified with linear subspaces in the (x, y)-plane and can thus be parameterized by an angle $\theta \in [0, \pi)$ in the obvious way. With respect to this identification the almost sure dimension estimates from [2] can be summarized as follows: *Given a Borel set* $A \subset \mathbb{H}^1$, *the following dimension estimates hold:*

$$\max\{0, \min\{\dim_H A - 2, 1\}\} \leq \dim P_{\mathbb{V}_\theta} A \leq \min\{\dim_H A, 1\}$$

and

$$\max\{\min\{\dim_H A, 1\}, 2\dim_H A - 5\} \leq \dim P_{\mathbb{V}_\theta^\perp} A \leq \min\left\{2\dim_H A, \frac{1}{2}(\dim_H A + 3), 3\right\}$$

for almost every $\theta \in [0, \pi)$.

One goal of the present paper is to establish similar results on higher-dimensional Heisenberg groups. In this case there exist not only horizontal lines, but also higher-dimensional horizontal subgroups which cannot be so easily parameterized. Thus, if we want to formulate almost sure dimension estimates for projection to horizontal or complementary vertical subgroups, we first need a natural measure on the space of all m-dimensional horizontal subgroups of $\mathbb{H}^n$ analogous to the measure $\gamma_{n,m}$ on the Grassmannian $G(n, m)$. Since not all linear subspaces V of $\mathbb{R}^{2n}$ correspond to horizontal subgroups $\mathbb{V}$, we cannot work with the full Grassmannian $G(2n, m)$. Instead we employ the so-called *isotropic Grassmannian* which is defined as the space

$$G_h(n, m) := \{V \in G(2n, m) : \ V \text{ an isotropic subspace of } \mathbb{R}^{2n}\}.$$

To each $V \in G_h(n, m)$ we associate a horizontal subgroup $\mathbb{V}$ and a complementary vertical subgroup $\mathbb{V}^\perp$ as above. Similarly as one defines the natural measure $\gamma_{n,m}$ on $G(n, m)$ starting from the Haar measure on the orthogonal group $O(n)$, one can construct a measure $\mu_{n,m}$ on $G_h(n, m)$ from Haar measure on the unitary group $U(n)$. It is with respect to this measure that our results are formulated. We emphasize the elementary but important fact that the isotropic Grassmannian $G_h(n, m)$ is a submanifold of $G(2n, m)$ of positive codimension, and that the measure $\mu_{n,m}$ does **not** coincide with the restriction of $\gamma_{2n,m}$ to $G_h(n, m)$.

Throughout the paper we assume that m and n are integers with $1 \leq m \leq n$. For two expressions A and B, we will write

$$A \lesssim B$$

if there exists a constant C such that $A \leq CB$; the dependence of parameters like $m, n, s, \ldots,$ will be clear from the context.

For the horizontal projections we obtain the following dimension estimates, which are exact generalizations of results which hold in the case $n = 1$.

Theorem 1.1. *Let* $A \subset \mathbb{H}^n$ *be a Borel subset. Then*

$$\dim P_{\mathbb{V}} A \leq \min\{\dim_H A, m\} \quad \textit{for all } V \in G_h(n, m)$$

and

$$\dim P_{\mathbb{V}} A \geq \max\{0, \min\{\dim_H A - 2, m\}\} \quad \textit{for } \mu_{n,m} \textit{ almost all } V \in G_h(n, m).$$

Furthermore, if $\dim_H A > m + 2$ *then* $\mathcal{H}^m(P_{\mathbb{V}} A) > 0$ *for* $\mu_{n,m}$ *a.e.* V.

In the proof of Theorem 1.1 we will use the following purely Euclidean result on the dimension of projections onto isotropic subspaces of $\mathbb{R}^{2n}$. This result may also have applications in symplectic geometry and seems to be of independent interest.

Theorem 1.2. *If A is a Borel set in $\mathbb{R}^{2n}$, then*

$$\dim_E \pi_V(A) = \min\{\dim_E A, m\} \tag{1.3}$$

for $\mu_{n,m}$ almost every $V \in G_h(n,m)$.

As mentioned before, the isotropic Grassmannian $G_h(n,m)$ is a submanifold of $G(2n,m)$. The $\mu_{n,m}$-almost sure estimate in Theorem 1.2 cannot be derived from known almost sure dimension estimates for the usual Grassmannian $G(2n,m)$, even if one takes into account the more precise statements on the Hausdorff dimension of exceptional sets as in [15]. See Remark 3.3 for further discussion.

Peres and Schlag [19] made a deep study of the measure-theoretic properties of nonlinear projection-type mappings. One reason for Theorem 1.2 to hold is that the projections π_V, $V \in G_h(n,m)$, satisfy the transversality condition of Peres and Schlag. This has been shown by Hovila [11]. Using results from [19] Hovila has obtained an alternate proof of Theorem 1.2 as well as an estimate for the Hausdorff dimension of the set of exceptional projections.

Let us briefly remark that Theorems 1.1 and 1.2, as well as our later Theorems 1.3 and 1.5 hold more generally for the class of so-called *Suslin sets*. The derivation of Theorem 1.1 from Theorem 1.2, even for Borel sets $A \subset \mathbb{H}^n$, requires knowledge of the latter for Suslin sets.

The situation is more subtle for the vertical projections. In this paper we limit ourselves to the discussion of dimension bounds for vertical projections of low dimensional sets. There, a sharp lower bound can be obtained by potential theoretic methods, using energy integrals. Although the approach is the same as in the first Heisenberg group, the proof is more difficult as it is more subtle to bound certain integrals, which are now given with respect to the measure $\mu_{n,m}$. We obtain the following result.

Theorem 1.3. *Let $A \subset \mathbb{H}^n$ be a Borel subset with $\dim_H A \le 1$. Then*

$$\dim_H P_{\mathbb{V}^\perp} A \le 2 \dim_H A \quad \text{for all } V \in G_h(n,m)$$

and

$$\dim_H P_{\mathbb{V}^\perp} A \ge \dim_H A \quad \text{for } \mu_{n,m} \text{ almost every } V \in G_h(n,m).$$

The universal upper bound follows easily from the local $\frac{1}{2}$-Hölder continuity of $P_{\mathbb{V}^\perp}$. Both the upper and lower bounds indicated in the theorem are sharp. To show sharpness of the upper bound, let A be a subset of the x_1-axis with $\dim_H A \in [0,1]$ prescribed. To show sharpness of the lower bound, let A be a subset of the t-axis with $\dim_H A \in [0,1]$ prescribed.

In the first Heisenberg group there are also sharp universal dimension estimates which hold for *all* vertical subgroups, see [2]. Such results can be proved using suitable covering arguments and the comparison of Hausdorff dimensions with respect to the Euclidean and Heisenberg metric. The *Dimension Comparison Principle* in the Heisenberg group asserts that for any set $A \subset \mathbb{H}^n$ with dimensions $\dim_E A = \alpha \in [0, 2n+1]$ and $\dim_H A = \beta \in [0, 2n+2]$, the inequalities

$$\max\{\alpha, 2\alpha - 2n\} =: \beta_-(\alpha) \le \beta \le \beta_+(\alpha) := \min\{2\alpha, \alpha + 1\} \tag{1.4}$$

hold true. For more information on the Dimension Comparison Principle we refer the reader to [3] and [4]. Similar ideas can be used to prove improved dimension bounds for sets A lying in a vertical subgroup $\mathbb{W} = \mathbb{V}^\perp$ with $V \in G_h(n,m)$. In this case, less horizontal directions have to be considered. We leave it to the reader to verify that in this case, one has

$$\max\{\alpha, 2\alpha - (2n-m)\} =: \beta_-^{\mathbb{W}}(\alpha) \le \beta \le \beta_+^{\mathbb{W}}(\alpha) := \min\{2\alpha, \alpha+1\}. \tag{1.5}$$

Using the estimates in (1.4) and (1.5) and similar techniques as in [2], we establish the following universal dimension estimates for projections in higher-dimensional Heisenberg groups.

Theorem 1.4. *Let $A \subset \mathbb{H}^n$ be a Borel set, let $V \in G_h(n,m)$, and let $\mathbb{W}$ be the complementary vertical subgroup associated to the homogeneous horizontal subgroup $\mathbb{V} = V \times \{0\} \subset \mathbb{H}^n$. Denoting $s = \dim_H A$ and $t = \dim_H P_{\mathbb{W}}(A)$ we have*

$$\max\left\{0, \frac{1}{2}(s-m), s-m-1, 2(s-n-1)-m\right\}$$
$$\le t \le \min\left\{2s, s+1, \frac{1}{2}(s-m)+n+1, 2n+2-m\right\}. \tag{1.6}$$

We believe that the estimates in (1.6) are sharp for all n and m, but we do not have explicit examples to this effect at present.

The last part of this paper is devoted to the study of the Hausdorff dimension of sets intersected with cosets of vertical subgroups. As mentioned above, this kind of result is related to the projection theorems and uses similar machinery from geometric measure theory. We will prove the following analogue of the Euclidean *slicing* (or *intersection*) *theorem*.

Theorem 1.5. *Let $A \subset \mathbb{H}^n$ be a Borel set with $\dim_H A > m+2$ such that $0 < \mathcal{H}_H^{\dim_H A}(A) < \infty$. Then $\mathcal{H}^m(\{u \in \mathbb{V}: \dim_H(A \cap (\mathbb{V}^\perp * u)) = \dim_H A - m\}) > 0$ for $\mu_{n,m}$-a.e. $V \in G_h(n,m)$.*

We are able to analyze the intersections with cosets of vertical subgroups, in part, because we have good information about projection to horizontal subgroups. Note that $\mathbb{V}^\perp * u = P_{\mathbb{V}}^{-1}(u)$ for any $u \in \mathbb{V}$. We have not been able to prove results concerning slicing with cosets of horizontal subgroups. One reason for this is that we do not have a sufficiently good understanding of the dimension distortion behavior of vertical projections.

Theorem 1.1 is sharp as we shall show in Remark 3.4. It is also easy to see by similar examples that the condition $\dim_H A > m+2$ is necessary in Theorem 1.5. As noted above Theorem 1.3 is sharp for sets of dimension at most one. However, sharp inequalities for vertical projections of sets of dimension bigger than one remain an open problem, even in $\mathbb{H}^1$ (cf. the discussion in the introduction of [2]).

The structure of this paper is as follows. We start in Section 2 with the definition of the isotropic Grassmannian and a discussion of its properties. In Section 3 we discuss dimension bounds for horizontal projections. In particular, we prove Theorems 1.1 and 1.2. Estimates for vertical projections on higher-dimensional Heisenberg groups are discussed in Section 4. In particular, we prove Theorem 1.3 in that section. In Section 5, we prove universal dimension estimates for vertical projections. In the final section (Section 6) we prove Theorem 1.5 and discuss other Heisenberg counterparts of Euclidean slicing theorems.

2. Isotropic Grassmannians

In this section we introduce the isotropic Grassmannian which provides the appropriate parameter space for projection and slicing theorems in the Heisenberg group. We discuss these Grassmannians as metric measure spaces and also as homogeneous spaces.

Fix integers $1 \le m \le n$. We introduce the *isotropic Grassmannian*

$$G_h(n,m) := \{V \in G(2n,m) :\ V \text{ isotropic subspace of } \mathbb{R}^{2n}\}$$

comprising all linear subspaces with the property that the symplectic form

$$\omega(z,z') = \sum_{i=1}^{n} y_i x_i' - y_i' x_i = \langle \mathbf{J}z, z' \rangle$$

with

$$z = (x_1,\dots,x_n,y_1,\dots,y_n), \qquad z' = (x_1',\dots,x_n',y_1',\dots,y_n') \in \mathbb{R}^{2n}$$

vanishes on V. Here we denoted by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^{2n}$ and by

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix}.$$

The space $G_h(n,n)$ is called the *Lagrangian Grassmannian* and is well-known in the literature [1,6]. The isotropic Grassmannians were previously considered in the context of Heisenberg geometry by Mattila et al. [17].

There is a one-to-one correspondence between elements $V \in G_h(n,m)$ and horizontal homogeneous subgroups $\mathbb{V} = V \times \{0\}$ of $\mathbb{H}^n$. To illustrate this fact, we make the following observation. Assume that a homogeneous subgroup $\mathbb{V}$ of $\mathbb{H}^n$ is completely contained inside the hyperplane $t = 0$. Since $\mathbb{V}$ is by definition closed under group multiplication, for any $(x,y,0)$ and $(x',y',0)$ in $\mathbb{V}$, we have that

$$(x,y,0) * (x',y',0) = \left(x + x', y + y', 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i')\right)$$

is an element of $\mathbb{V}$ and therefore necessarily, $\sum_{i=1}^{n}(y_i x_i' - x_i y_i') = 0$. It is not hard to see that this condition is also sufficient for linear subspaces of $\mathbb{R}^{2n+1}$ contained in the hyperplane $t = 0$ in order to carry the structure of a homogeneous subgroup.

In the first Heisenberg group the space of non-trivial horizontal subgroups

$$\mathbb{V}_\theta = \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\} \times \{0\}, \quad \theta \in [0,\pi)$$

can be equipped in a natural way with the Lebesgue measure on $[0,\pi)$. The situation becomes more complicated in a higher dimensional Heisenberg group $\mathbb{H}^n$ with $n > 1$. There, one can endow $G_h(n,m)$ with a natural measure $\mu_{n,m}$ in a similar way as $G(n,m)$ is endowed with the measure $\gamma_{n,m}$ (see [16, Chapter 3]), using unitary instead of orthogonal matrices.

Recall that a matrix $\mathbf{C} \in M(2n,\mathbb{R})$ is called *orthogonal*, written $\mathbf{C} \in O(2n,\mathbb{R})$, if $\mathbf{C}^T\mathbf{C} = \mathbf{C}\mathbf{C}^T = \mathbf{I}_{2n}$. The corresponding linear map preserves the standard inner product, or equivalently, the Euclidean distance on $\mathbb{R}^{2n}$.

A matrix $\mathbf{C} \in M(2n,\mathbb{R})$ is said to be *symplectic*, written $\mathbf{C} \in \mathrm{Sp}(n)$, if it preserves the symplectic form: $\omega(\mathbf{C}z,\mathbf{C}z') = \omega(z,z')$ for all $z,z' \in \mathbb{R}^{2n}$. Equivalently, $\mathbf{C}^T\mathbf{J}\mathbf{C} = \mathbf{J}$.

Next we consider complex linear maps. A matrix $\mathbf{C} \in M(n,\mathbb{C})$ is called *unitary*, written $\mathbf{C} \in U(n,\mathbb{C})$, if $\mathbf{C}^*\mathbf{C} = \mathbf{C}\mathbf{C}^* = \mathbf{I}_n$, where $\mathbf{C}^* = \overline{\mathbf{C}}^T$ denotes the adjoint of $\mathbf{C}$. Equivalently, $\mathbf{C}$ preserves the Hermitian scalar product on $\mathbb{C}^n$.

Each $\mathbf{C} \in M(n, \mathbb{C})$ can be written uniquely as $\mathbf{C} = \mathbf{A} + \mathbf{iB}$ with $\mathbf{A}, \mathbf{B} \in M(n, \mathbb{R})$. Now $U(n, \mathbb{C})$ can be identified with a subgroup of $M(2n, \mathbb{R})$ through the mapping

$$M(n, \mathbb{C}) \to M(2n, \mathbb{R}), \qquad \mathbf{A} + \mathbf{iB} \mapsto \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}.$$

This is a monomorphism which maps $U(n, \mathbb{C})$ on the subgroup

$$U(n) := \mathrm{Sp}(n) \cap O(2n, \mathbb{R}),$$

see for instance Proposition 2.12 in [9].

We will denote by $O(2n) = O(2n, \mathbb{R})$, $\mathrm{Sp}(n)$, $U(n)$ etc. both the above defined matrix groups and the sets of linear maps which have a corresponding matrix representation.

The orthogonal group $O(n)$ acts transitively on the unit sphere $\mathbb{S}^{n-1}$ and on the Grassmannian $G(n, m)$. In a similar fashion, $U(n, \mathbb{C})$ acts transitively on $\mathbb{S}^{2n-1}$, the unit sphere in $\mathbb{C}^n$, and on the isotropic Grassmannian $G_h(n, m)$. Note that $gV_0 \in G_h(n, m)$ for all $V_0 \in G_h(n, m)$ and $g \in U(n)$. Indeed, since V_0 is isotropic by assumption, the symplectic form ω vanishes identically on V_0. Since $g \in U(n)$ is symplectic, ω vanishes also on $g(V_0)$.

Lemma 2.1. *The group $U(n)$ acts transitively on $\mathbb{S}^{2n-1}$ and on $G_h(n, m)$.*

In the proof of this lemma the same ideas are used as, for instance, in the proof of Theorem 1.26 (i) in [9].

Proof. Let V and V' be two isotropic m-dimensional subspaces of $\mathbb{R}^{2n}$ with orthonormal bases $\mathcal{E} = \{e_1, \ldots, e_m\}$ and $\mathcal{E}' = \{e'_1, \ldots, e'_m\}$ respectively. Any isotropic subspace is contained in a Lagrangian one, see for instance [9, p.15]. Denote by W and W' two such Lagrangian spaces which contain V and V', respectively. By completing the set $\mathcal{E}$ to a basis of W and applying Gram–Schmidt to the added vectors, we can find vectors $e_{m+1}, \ldots, e_n$ such that

$$\mathcal{O} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$$

is an orthonormal basis of W. Analogously, one can find $\{e'_{m+1}, \ldots, e'_n\}$ such that

$$\mathcal{O}' = \{e'_1, \ldots, e'_m, e'_{m+1}, \ldots, e'_n\}$$

is an orthonormal basis of W'. Then $\mathcal{B} = \mathcal{O} \cup \mathbf{J}\mathcal{O}$ and $\mathcal{B}' = \mathcal{O}' \cup \mathbf{J}\mathcal{O}'$ are *orthosymplectic* bases of $\mathbb{R}^{2n}$, this means that they are bases which are both symplectic, i.e.

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \qquad \omega(f_i, e_j) = \delta_{i,j} \quad \text{for } i, j \in \{1, \ldots, n\}$$

(where $f_i = -\mathbf{J}e_i$) and orthogonal with respect to the standard scalar product on $\mathbb{R}^{2n}$. Then there exists $U \in O(2n)$ with $U(e_i) = e'_i$ and $U(f_i) = f'_i$ for all $i \in \{1, \ldots, n\}$. In particular, U maps $\mathcal{E}$ to $\mathcal{E}'$ and thus maps the isotropic subspace V to V' as desired.

Let $z, \tilde{z} \in \mathbb{R}^{2n}$ be two arbitrary points, which we write in the form $z = \sum_{i=1}^n a_i e_i + b_i f_i$ and $\tilde{z} = \sum_{i=1}^n \tilde{a}_i e_i + \tilde{b}_i f_i$. Since the bases $\mathcal{O}$ and $\mathcal{O}'$ are orthosymplectic, it follows that

$$\omega(U(z), U(\tilde{z})) = \sum_{i=1}^n \tilde{a}_i b_i - a_i \tilde{b}_i = \omega(z, \tilde{z}).$$

Hence $U \in \mathrm{Sp}(n)$ and therefore $U \in U(n) = \mathrm{Sp}(n) \cap O(2n)$ as desired. In particular, the proof for $m = 1$ shows that $U(n)$ acts transitively on $\mathbb{S}^{2n-1}$. $\square$

Now fix $V_0 \in G_h(n,m)$ and let ϑ_n be Haar measure on the group $U(n)$ with $\vartheta_n(U(n)) = 1$. We define a Radon probability measure $\mu_{n,m} = f_{V_0\sharp}\vartheta_n$ with $f_{V_0}(g) = gV_0$, i.e.

$$\mu_{n,m}(A) := \vartheta_n(\{g \in U(n):\ gV_0 \in A\}) \quad \text{for } A \subset G_h(n,m). \tag{2.1}$$

We will show that $\mu_{n,m}$ does not depend on the choice of V_0. We claim that $\mu_{n,m}$ is invariant under $U(n)$. More precisely, for any $g \in U(n)$ and $A \subset G_h(n,m)$, one has

$$\mu_{n,m}(gA) = \mu_{n,m}(A).$$

This follows from the fact that ϑ_n is a Haar measure on $U(n)$:

$$\begin{aligned} \mu_{n,m}(gA) &= \vartheta_n(\{h \in U(n):\ hV_0 \in gA\}) = \vartheta_n(\{h \in U(n):\ g^{-1}hV_0 \in A\}) \\ &= \vartheta_n(\{g^{-1}h \in U(n):\ g^{-1}hV_0 \in A\}) = \vartheta_n(\{h \in U(n):\ hV_0 \in A\}) \\ &= \mu_{n,m}(A). \end{aligned}$$

We define

$$d(V,V') := \|\pi_V - \pi_{V'}\| \quad \text{for } V, V' \in G_h(n,m), \tag{2.2}$$

where $\|\cdot\|$ denotes the usual operator norm for linear maps and $\pi_V : \mathbb{R}^{2n} \to V$ denotes the usual Euclidean orthogonal projection. This yields a metric on $G_h(n,m)$. The group $U(n) \subset O(2n)$ acts on $G_h(n,m) \subset G(2n,m)$ by isometries of this metric.

Consider now an arbitrary $U(n)$ invariant Radon measure μ on $G_h(n,m)$. Let $V, V' \in G_h(n,m)$ and $0 < r < \infty$. Since $U(n)$ acts transitively on $G_h(n,m)$ there exists $g \in U(n)$ such that $gV = V'$. We exploit the fact that the distance d on $G_h(n,m)$ is preserved under the action of $U(n)$ and the measure μ is by assumption $U(n)$ invariant. This yields

$$\begin{aligned} \mu(B(V',r)) &= \mu(B(gV,r)) = \mu(\{\tilde V:\ d(\tilde V, gV) \le r\}) = \mu(\{\tilde V:\ d(g^{-1}\tilde V, V) \le r\}) \\ &= \mu(\{g^{-1}\tilde V:\ d(g^{-1}\tilde V, V) \le r\}) = \mu(\{\tilde V:\ d(\tilde V, V) \le r\}) = \mu(B(V,r)). \end{aligned}$$

We conclude that every $U(n)$ invariant Radon measure μ on $G_h(n,m)$ is *uniformly distributed*:

$$0 < \mu(B(V,r)) = \mu(B(V',r)) < \infty \quad \text{for all } V, V' \in G_h(n,m),\ 0 < r < \infty.$$

Lemma 2.2. *The measures $f_{V_0\sharp}\vartheta_n$, $V_0 \in G_h(n,m)$, are all equal.*

Proof. As discussed above, the measures $f_{V_0\sharp}\vartheta_n$, $V_0 \in G_h(n,m)$, are $U(n)$ invariant, uniformly distributed probability measures. Hence they are all equal, see [16, Thm 3.4]. □

We fix once and for all

$$V_0 := \{z = (x,y) \in \mathbb{R}^{2n}:\ x_{m+1} = \cdots = x_n = y_1 = \cdots = y_n = 0\}. \tag{2.3}$$

The following result is the analogue of Theorem 3.7 in [16] for $U(n)$ instead of $O(n)$.

Theorem 2.3. *For $z \in \mathbb{S}^{2n-1}$ and $A \subset \mathbb{S}^{2n-1}$, $\vartheta_n(\{g \in U(n):\ g(z) \in A\}) = \sigma^{2n-1}(A)$, where σ^{2n-1} denotes normalized surface measure on $\mathbb{S}^{2n-1}$.*

Proof. Note that $\vartheta_n(\{g \in U(n):\ g(z) \in A\}) = (f_{z\sharp}\vartheta_n)(A)$ where $f_z : U(n) \to \mathbb{S}^{2n-1}$ is defined by $f_z(g) := g(z)$. Our goal is to show $f_{z\sharp}\vartheta_n = \sigma^{2n-1}$. To this end, we first observe that $\sigma^{2n-1}(\mathbb{S}^{2n-1}) = 1 = \vartheta_n(U(n)) = f_{z\sharp}\vartheta_n(\mathbb{S}^{2n-1})$. Since σ^{2n-1} is uniformly distributed on $\mathbb{S}^{2n-1}$, the equality of these two measures will follow if we show that $f_{z\sharp}\vartheta_n$ is also uniformly distributed on $\mathbb{S}^{2n-1}$. This can be seen as explained before, using the fact that $U(n)$ acts transitively on $\mathbb{S}^{2n-1}$ and that ϑ_n is $U(n)$ invariant by definition. □

The following (purely Euclidean) Lemma is crucial for dimension bounds for both horizontal and vertical projections.

Lemma 2.4. *There exists $C = C(n)$ such that for all $z \in \mathbb{R}^{2n} \setminus \{0\}$ and $0 < \delta < \infty$, one has*

$$\mu_{n,m}(\{V \in G_h(n,m) : |\pi_V z| \le \delta\}) \le C\delta^m |z|^{-m} \tag{2.4}$$

and

$$\mu_{n,m}(\{V \in G_h(n,m) : |\pi_{V^\perp} z| \le \delta\}) \le C\delta^{2n-m} |z|^{m-2n}. \tag{2.5}$$

Proof. The proof is analogous to that of Lemma 3.11 in [16], using Theorem 2.3. First, we explain how to prove (2.4). Second, we sketch how to derive (2.5) by a similar argument.

Notice that

$$\pi_V z = |z|\pi_V\left(\frac{z}{|z|}\right)$$

which allows us to assume $|z| = 1$. Recall that we are working with the Euclidean distance and the Euclidean orthogonal projection on V. Therefore $|\pi_V z| = \operatorname{dist}(z, V^\perp)$. Consequently

$$\begin{aligned}\mu_{n,m}(\{V : |\pi_V z| \le \delta\}) &= \mu_{n,m}(\{V : \operatorname{dist}(z, V^\perp) \le \delta\}) \\ &= \vartheta_n(\{g \in U(n) : \operatorname{dist}(z, g(V_0)^\perp) \le \delta\}),\end{aligned}$$

where V_0 is as in (2.3); the second equality here follows directly from the definition of the measure $\mu_{n,m}$. Since each $g \in U(n)$ is orthogonal we have $g(V_0)^\perp = g(V_0^\perp)$ and therefore

$$\begin{aligned}\mu_{n,m}(\{V : |\pi_V z| \le \delta\}) &= \vartheta_n(\{g \in U(n) : \operatorname{dist}(z, g(V_0^\perp)) \le \delta\}) \\ &= \vartheta_n(\{g \in U(n) : \operatorname{dist}(g^{-1}(z), V_0^\perp) \le \delta\}) \\ &= \vartheta_n(\{g \in U(n) : \operatorname{dist}(g(z), V_0^\perp) \le \delta\}) \\ &= \sigma^{2n-1}(\{w \in \mathbb{S}^{2n-1} : \operatorname{dist}(w, V_0^\perp) \le \delta\}),\end{aligned}$$

where we have applied Theorem 2.3 in the last step. Then it follows as in [16, p.50] that

$$\sigma^{2n-1}(\{w \in \mathbb{S}^{2n-1} : \operatorname{dist}(w, V_0^\perp) \le \delta\}) \le \alpha(2n)^{-1}2^{2n}\delta^m,$$

which concludes the proof of (2.4).

The second inequality (2.5) can be derived by a similar argument. First, one proves that

$$\mu_{n,m}(\{V : |\pi_{V^\perp} z| \le \delta\}) = \sigma^{2n-1}(\{w \in \mathbb{S}^{2n-1} : \operatorname{dist}(w, V_0) \le \delta\}).$$

We conclude as before that $\sigma^{2n-1}(\{w \in \mathbb{S}^{2n-1} : \operatorname{dist}(w, V_0) \le \delta\}) \le \alpha(2n)^{-1}2^{2n}\delta^{2n-m}$. □

The isotropic Grassmannian $G_h(n,m)$, $m \in \{1,\dots,n\}$, can also be endowed with the structure of a *homogeneous space*, using the transitive action of $U(n)$ on $G_h(n,m)$. In order to describe this space more precisely, we study the stabilizer subgroup

$$G_{V_0} = \{g \in U(n) : gV_0 = V_0\}$$

of the fixed subspace V_0 given in (2.3).

Lemma 2.5. *Let $h : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ be linear. The following conditions are equivalent:*

1. $h \in U(k)$;
2. $h = \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}$ *with* $\mathbf{A}, \mathbf{B} \in M(k, \mathbb{R})$, $\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t = \mathbf{I}_k$, *and* $\mathbf{B}\mathbf{A}^t = \mathbf{A}\mathbf{B}^t$;
3. $h = \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}$ *with* $\mathbf{A}, \mathbf{B} \in M(k, \mathbb{R})$, $\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} = \mathbf{I}_k$, *and* $\mathbf{B}^t\mathbf{A} = \mathbf{A}^t\mathbf{B}$.

This fact is standard, see e.g., [9, p. 33].

Proposition 2.6. *Let V_0 be as in (2.3). A linear transformation $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ belongs to the stabilizer group G_{V_0} if and only if it has the form*

$$g = \begin{pmatrix} \mathbf{G}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{G}_{22} & 0 & \mathbf{G}_{24} \\ 0 & 0 & \mathbf{G}_{33} & 0 \\ 0 & \mathbf{G}_{42} & 0 & \mathbf{G}_{44} \end{pmatrix}$$

with $\mathbf{G}_{11}, \mathbf{G}_{33} \in M(m, \mathbb{R})$, $\mathbf{G}_{22}, \mathbf{G}_{24}, \mathbf{G}_{42}, \mathbf{G}_{44} \in M(n-m, \mathbb{R})$, $\mathbf{G}_{11} = \mathbf{G}_{33} \in O(m)$ *and*

$$\begin{pmatrix} \mathbf{G}_{22} & \mathbf{G}_{24} \\ \mathbf{G}_{42} & \mathbf{G}_{44} \end{pmatrix} \in U(n-m). \tag{2.6}$$

The stabilizer group G_{V_0} is isomorphic to $O(m) \times U(n-m)$.

Proof. We decompose $\mathbb{R}^{2n}$ as $\mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^{n-m}$ and write accordingly a given linear map $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ in block matrix form

$$g = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} & \mathbf{G}_{14} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} & \mathbf{G}_{24} \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33} & \mathbf{G}_{34} \\ \mathbf{G}_{41} & \mathbf{G}_{42} & \mathbf{G}_{43} & \mathbf{G}_{44} \end{pmatrix}.$$

Since g preserves $\mathbb{R}^m \times \{0\}$, it follows that $\mathbf{G}_{21} = \mathbf{G}_{31} = \mathbf{G}_{41} = 0$. As g is unitary, we may apply Lemma 2.5 with

$$\mathbf{A} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ 0 & \mathbf{G}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{G}_{33} & \mathbf{G}_{34} \\ \mathbf{G}_{43} & \mathbf{G}_{44} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & \mathbf{G}_{32} \\ 0 & \mathbf{G}_{42} \end{pmatrix} = \begin{pmatrix} -\mathbf{G}_{13} & -\mathbf{G}_{14} \\ -\mathbf{G}_{23} & -\mathbf{G}_{24} \end{pmatrix}. \tag{2.7}$$

The condition $\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t = \mathbf{I}_n$ implies

$$\mathbf{G}_{22}\mathbf{G}_{22}^t + \mathbf{G}_{42}\mathbf{G}_{42}^t = \mathbf{I}_{n-m}, \tag{2.8}$$

while $\mathbf{B}\mathbf{A}^t = \mathbf{A}\mathbf{B}^t$ yields

$$\mathbf{G}_{42}\mathbf{G}_{22}^t = \mathbf{G}_{22}\mathbf{G}_{42}^t. \tag{2.9}$$

With the help of Lemma 2.5 it follows from (2.7), (2.8) and (2.9) that (2.6) holds. Using $\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} = \mathbf{I}_n$, we find $\mathbf{G}_{11}^t\mathbf{G}_{11} = \mathbf{I}_m$, which suffices to show that $\mathbf{G}_{11} \in O(m)$ and thus, by (2.7), also $\mathbf{G}_{33} \in O(m)$. Moreover, (2.7) immediately implies $\mathbf{G}_{13} = \mathbf{G}_{23} = \mathbf{G}_{43} = 0$. The condition $\mathbf{A}^t\mathbf{A} + \mathbf{B}^t\mathbf{B} = \mathbf{I}_n$ also yields $\mathbf{G}_{11}^t\mathbf{G}_{12} = 0$, which gives $\mathbf{G}_{12} = \mathbf{G}_{34} = 0$. Similarly, one can conclude from $\mathbf{B}^t\mathbf{A} = \mathbf{A}^t\mathbf{B}$ that $\mathbf{G}_{32} = \mathbf{G}_{14} = 0$. It follows that the matrix corresponding to g is of the desired form.

Conversely, it is easily verified that any block matrix of this form belongs to G_{V_0}.

Finally, the map from $O(m) \times U(n-m)$ to $U(n)$ given by

$$\left(\mathbf{G}_{11}, \begin{pmatrix} \mathbf{G}_{22} & \mathbf{G}_{24} \\ \mathbf{G}_{42} & \mathbf{G}_{44} \end{pmatrix}\right) \mapsto \begin{pmatrix} \mathbf{G}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{G}_{22} & 0 & \mathbf{G}_{24} \\ 0 & 0 & \mathbf{G}_{11} & 0 \\ 0 & \mathbf{G}_{42} & 0 & \mathbf{G}_{44} \end{pmatrix}$$

is an isomorphism from $O(m) \times U(n-m)$ onto G_{V_0}. □

Proposition 2.7. *The isotropic Grassmannian $G_h(n,m)$ is isomorphic to the quotient space $U(n)/O(m) \times U(n-m)$.*

Proof. The group $U(n)$ acts transitively on $G_h(n,m)$. By a well-known result, $G_h(n,m)$ is isomorphic to $U(n)/G_{V_0}$. The claim follows by Proposition 2.6. □

In the following, we discuss in more detail the homogeneous space $U(n)/O(m) \times U(n-m)$; this allows us to better understand the Grassmannian $G_h(n,m)$.

As mentioned above, we identify $G_h(n,m)$ with $U(n)/O(m) \times U(n-m)$ by the mapping

$$V = gV_0 \quad \longleftrightarrow \quad gG_{V_0}.$$

The identification is well defined since $gV = g'V$ for $g, g' \in U(n)$ if and only if $gG_{V_0} = g'G_{V_0}$. The quotient $U(n)/O(m) \times U(n-m)$ carries the structure of a smooth manifold with dimension

$$\begin{aligned} \dim U(n)/O(m) \times U(n-m) &= \dim U(n) - \dim O(m) \times U(n-m) \\ &= n^2 - \left(\frac{m(m-1)}{2} + (n-m)^2\right) \\ &= 2nm - \frac{m(3m-1)}{2}. \end{aligned}$$

In particular, the Lagrangian Grassmannian $G_h(n,n)$ is a manifold of dimension $\frac{1}{2}n(n+1)$.

Remark 2.8. The codimension of $G_h(n,m)$ in $G(2n,m)$ is

$$(2nm - m^2) - \left(2nm - \frac{m(3m-1)}{2}\right) = \frac{m(m-1)}{2} = \dim O(m).$$

This can be explained as follows: there is an extra $O(m)$ degree of freedom in $G(2n,m)$ which is not present in $G_h(2n,m)$ since in $G(2n,m)$, unlike in $G_h(n,m)$, we are free to rotate the symplectic complement of the given subspace V by an orthogonal map.

The standard invariant Riemannian metric on $U(n)$ induces a Riemannian metric and a Riemannian volume on the homogeneous space $G_h(n,m)$. The resulting metric is bi-Lipschitz equivalent to the metric introduced in (2.2), while the resulting measure is comparable to the Hausdorff measure of dimension $\dim G_h(n,m)$ and to $\mu_{n,m}$.

Remark 2.9. To conclude this section we remark that $U(n)$ also acts by isometries on the Heisenberg group $(\mathbb{H}^n, d_H)$, according to the formula $g \cdot (z,t) = (gz,t)$. In the following section we will consider the images of subsets of both $\mathbb{R}^{2n}$ and $\mathbb{H}^n$ by unitary matrices.

3. Dimension bounds for horizontal projections

In this section we discuss upper and lower bounds for the dimension of horizontal projections of subsets of the Heisenberg group $\mathbb{H}^n$. In particular, we prove Theorem 1.1.

We begin by establishing Theorem 1.2. As mentioned in the introduction, this purely Euclidean result may be of independent interest.

The proof of Theorem 1.2 uses energy estimates and Frostman's lemma. For Suslin subsets of complete, separable metric spaces, such results are due to Howroyd, see [12]. We briefly recall the relevant statements and refer the reader to [12] or [16, Chapter 8] for more details.

By $\mathcal{M}(A)$ we denote the collection of positive, finite Borel regular measures supported on a set A in a metric space X.

Theorem 3.1 (*Frostman's Lemma*). *Let A be a Suslin subset of a complete, separable metric space (X,d). Suppose that there exists $s>0$, $\mu\in\mathcal{M}(A)$, and $r_0\in(0,\infty]$ so that the inequality*

$$\mu(B(x,r))\le r^s \tag{3.1}$$

holds for all $x\in A$ and $0<r<r_0$. Then $\mathcal{H}^s(A)>0$. In particular, $\dim A\ge s$. Conversely, if $\mathcal{H}^s(A)>0$ then there exists $\mu\in\mathcal{M}(A)$ so that (3.1) holds for all $x\in A$ and $r>0$.

The *s-energy* of a measure $\mu\in\mathcal{M}(A)$ is defined to be

$$I_s(\mu)=\int_X\int_X d(x,y)^{-s}\,\mathrm{d}\mu x\,\mathrm{d}\mu y.$$

Theorem 3.2 (*Frostman's Lemma, Energy Version*). *Let A be a Suslin subset of a complete, separable metric space (X,d) and let $s>0$ be such that there exists $\mu\in\mathcal{M}(A)$ with $I_s(\mu)<\infty$. Then $\dim A\ge s$. Conversely, if $s<\dim A$, then there exists $\mu\in\mathcal{M}(A)$ with $I_s(\mu)<\infty$.*

Proof of Theorem 1.2. Let $A\subset\mathbb{R}^{2n}$ be a Suslin set. Since the upper bound in (1.3) holds trivially for all $V\in G_h(n,m)$, it suffices to establish the lower bound for $\mu_{n,m}$-a.e. V.

Let us assume that $\dim_E A\le m$. Pick an arbitrary $0<s<\dim_E A$. Then there exists $\mu\in\mathcal{M}(A)$ with

$$I_s(\mu)=\int_A\int_A |z-w|^{-s}\,\mathrm{d}\mu z\,\mathrm{d}\mu w<\infty.$$

Then, analogously as in the proof for $G(n,m)$, see [7] or [16],

$$\begin{aligned}\int_{G_h(n,m)} I_s(\pi_{V\sharp}\mu)\,\mathrm{d}\mu_{n,m}V &= \int_{G_h(n,m)}\int_A\int_A |\pi_V(z-w)|^{-s}\,\mathrm{d}\mu z\,\mathrm{d}\mu w\,\mathrm{d}\mu_{n,m}V\\ &= \int_A\int_A\int_{G_h(n,m)} |\pi_V(z-w)|^{-s}\,\mathrm{d}\mu_{n,m}V\,\mathrm{d}\mu z\,\mathrm{d}\mu w\lesssim I_s(\mu).\end{aligned}$$

In the last step we have used Lemma 2.4 and the following estimates:

$$\begin{aligned}&\int_{G_h(n,m)} |\pi_V(z-w)|^{-s}\,\mathrm{d}\mu_{n,m}V=\int_0^\infty \mu_{n,m}\big(\{V:\ |\pi_V(z-w)|\le\eta^{-\frac{1}{s}}\}\big)\,\mathrm{d}\eta\\ &\quad\le\int_0^{|z-w|^{-s}}\mathrm{d}\eta+\int_{|z-w|^{-s}}^\infty \mu_{n,m}\big(\{V:\ |\pi_V(z-w)|\le\eta^{-\frac{1}{s}}\}\big)\,\mathrm{d}\eta\end{aligned}$$

$$\le |z-w|^{-s} + C|z-w|^{-m}\int_{|z-w|^{-s}}^{\infty} \eta^{-\frac{m}{s}}\,\mathrm{d}\eta \le C|z-w|^{-s}.$$

Thus $I_s(\pi_{V\sharp}\mu)$ is finite for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$. By Theorem 3.2, we conclude that $\dim_E \pi_V(A) \ge s$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$. This completes the proof.

Remark 3.3. Theorem 1.2 is the analogue for $G_h(n,m)$ of classical Euclidean projection theorems for the usual Grassmannian. The result is somewhat surprising. Consider for example the case $n = m = 2$. Let A be a subset of $\mathbb{R}^4$ with $\dim_E A = s \le 2$. Then, by classical projection theorems, $\dim_E \pi_V(A) = \dim_E A$ for $\gamma_{4,2}$ a.e. $V \in G(4,2)$. By a result of Mattila [15] the exceptional set of spaces $V \in G(4,2)$ for which $\dim_E P_V(A) \neq \dim_E A$ is of dimension at most $2+s$, and by results of Mattila and Kaufman [13] this estimate is sharp. On the other hand, the dimension of the isotropic Grassmannian $G_h(2,2)$ is equal to 3 which is smaller than the largest possible dimension $2+s$ of the exceptional set related to A, provided that $\dim_E A = s > 1$. So one could a priori imagine a situation where $G_h(2,2)$, or more generally $G_h(n,m)$, is completely hidden inside the exceptional set of directions for which the lower dimension bound of Theorem 1.2 does not hold. The theorem shows that such a situation cannot occur.

By exploiting the relationship between the Euclidean projection π_V and the Heisenberg horizontal projection $P_{\mathbb{V}}$ with $\mathbb{V} = V \times \{0\}$, Theorem 1.2 can be applied in order to prove Theorem 1.1 on the almost sure dimension bounds for horizontal projections.

Proof of Theorem 1.1. The upper bound follows from the Lipschitz continuity of $P_{\mathbb{V}}$ and the monotonicity of the dimension, using the fact that the Heisenberg distance coincides with the Euclidean metric on every horizontal subgroup $\mathbb{V}$.

Concerning the lower bound, we prove by an appropriate covering argument that

$$\dim_E \pi(A) \ge \dim_H A - 2, \tag{3.2}$$

for every $A \subset \mathbb{H}^n$, where $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ denotes the standard projection $\pi(z,t) = z$.

We may assume without loss of generality that A is bounded. In fact, let us assume that $|t| \le 1$ for all points $p = (z,t) \in A$. Let $s > \dim_E \pi(A)$, let $\varepsilon > 0$, and cover the set $\pi(A)$ with a family of Euclidean balls $\{B_E(z_i,r_i)\}_i$ so that $\sum_i r_i^s < \varepsilon$. Since the projection map $\pi : (\mathbb{H}^n, d_H) \to (\mathbb{R}^{2n}, d_E)$ is 1-Lipschitz, the fiber $\pi^{-1}(B_E(z,r))$ contains the ball $B_H((z,t),r)$ for any $t \in \mathbb{R}$. The shape of Heisenberg balls is described by the well-known Ball–Box Theorem. We can choose an absolute constant $C_0 > 0$ and $N_i \le C_0 r_i^{-2}$ values t_{ij} so that the family $\{B_H((z_i,t_{ij}), C_0 r_i)\}_j$ covers the set $B_E(z_i,r_i) \times [-1,1]$, see also similar estimates in [4]. Then the family $\{B_H((z_i,t_{ij}), C_0 r_i)\}_{i,j}$ covers the set A. Denoting by $r(B)$ the radius of a ball B, we compute

$$\sum_{i,j} r\big(B_H((z_i,t_{ij}), C_0 r_i)\big)^{s+2} = \sum_i N_i (C_0 r_i)^{s+2} \le C \sum_i r_i^s \le C\varepsilon$$

for some C depending only on C_0 and s. Letting $\varepsilon \to 0$ gives

$$\mathcal{H}_H^{s+2}(A) = 0$$

so $\dim_H A \le s+2$. We obtain (3.2) by letting s tend to $\dim_E \pi(A)$.

Now we may apply Theorem 1.2 to the Suslin set $\pi(A)$ in order to get

$$\dim_H(P_{\mathbb{V}}(A)) = \dim_E \pi_V(\pi(A)) \ge \min\{\dim_E \pi(A), m\} \ge \min\{\dim_H A - 2, m\}$$

for $\mu_{n,m}$ almost every $V \in G_h(n,m)$. Note that in general the projected set $\pi(A)$ may be only a Suslin set, even if the original set $A \subset \mathbb{H}^n$ is a Borel set.

The final statement in the theorem (on the $\mu_{n,m}$ a.e. positivity of $\mathcal{H}^m(P_{\mathbb{V}}A)$) will follow from a result which we will prove in Section 6. See Proposition 6.1 and its Corollary 6.2. □

Remark 3.4. Theorem 1.1 is sharp as we will now demonstrate. For each $0 \le s \le m+2$ we will construct sets $A, B \subset \mathbb{H}^n$ so that $\dim_H A = \dim_H B = s$,

$$\dim P_{\mathbb{V}}(A) = \min\{s, m\} \tag{3.3}$$

for all $V \in G_h(n,m)$, and

$$\dim P_{\mathbb{V}}(B) = \max\{0, s-2\} \tag{3.4}$$

for all V in some subset of $G_h(n,m)$ of full $\mu_{n,m}$ measure. Moreover, for each $m+2 < s \le 2n+2$ we will construct a set $A \subset \mathbb{H}^n$ so that $\dim_H A = s$ and $\mathcal{H}^m(P_{\mathbb{V}}A) > 0$ for all $V \in G_h(n,m)$.

First we give examples realizing the upper bound; these are the sets denoted A in the previous paragraph. If $s \le m$, choose an s-dimensional set $A' \subset \mathbb{V}_0$, where V_0 is a fixed element of $G_h(n,m)$. Let $\{V_1, \dots, V_N\}$ be a $\frac{1}{2}$-net in $G_h(n,m)$. Corresponding to these subspaces, we may choose matrices $g_1, \dots, g_N \in U(n)$ with $g_i(V_0) = V_i$, $i = 1, \dots, N$. Let $A = \bigcup_{i=1}^N g_i(A')$. Then $\dim_H A = s$. Now consider $P_{\mathbb{V}}(A)$ for an arbitrary $V \in G_h(n,m)$. There exists $i \in \{1, \dots, N\}$ such that $d(V, V_i) < \frac{1}{2}$, where d denotes the metric on $G_h(n,m)$ defined as in (2.2). We claim that $V_i \cap V^{\perp} = \{0\}$. Indeed, assume towards a contradiction that there exists $v \in V_i \cap V^{\perp}$ with $v \ne 0$. Without loss of generality, we may assume that $|v| = 1$. It follows that

$$d(V, V_i) = \|\pi_V - \pi_{V_i}\| \ge |\pi_V(v) - \pi_{V_i}(v)| = |v| = 1,$$

which contradicts the assumption. Thus, $\ker(\pi_V|_{V_i}) = V_i \cap V^{\perp} = \{0\}$ and $\pi_V|_{V_i} : V_i \to V$ is bi-Lipschitz. Thus, $\dim_H P_{\mathbb{V}}(g_i(A')) = s$, in particular, $\dim_H P_{\mathbb{V}}(A) = s$.

If $s > m$, choose again any s-dimensional set $A' \subset \mathbb{H}^n$ and let $A = A' \cup \bigcup_{i=1}^N g_i(\mathbb{V}_0)$. Then $\dim_H A = s$ and $\mathcal{H}^m(P_{\mathbb{V}}(A)) > 0$ for all $V \in G_h(n,m)$.

We now give examples realizing the lower bound; these are the sets denoted B in the first paragraph of this remark. If $0 \le s \le 2$, choose an s-dimensional set B contained in the t-axis. Then $P_{\mathbb{V}}(B) = \{0\}$ for all $V \in G_h(n,m)$. It remains to construct the desired set B in case $2 \le s \le m+2$. For fixed $V_0 \in G_h(n,m)$, choose any set $B_1 \subset V_0$ of dimension $s-2$. Set $B = \{(z,t) : z \in B_1, t \in \mathbb{R}\} \subset \mathbb{H}^n$. Then B is s-dimensional and $P_{\mathbb{V}}(B) = \pi_V(B_1) \times \{0\}$ for any $V \in G_h(n,m)$. The result follows from Theorem 1.2.

We emphasize that the above construction works for an arbitrarily chosen set $B_1 \subset V_0$. On the other hand, it is possible to use the construction in the first part of the remark in order to define a particular set B with $\dim_H B = s$ and $\dim P_{\mathbb{V}}(B) = s-2$ for *all* $V \in G_h(n,m)$.

4. Lower bounds for the vertical projections

The aim of this section is to prove Theorem 1.3 on the lower dimension bound for vertical projections of low dimensional subsets in $\mathbb{H}^n$. Analogously as in the case $n = 1$, already discussed in [2], the idea is to use energy methods in order to establish this lower bound. The goal is to show for $0 < s < 1$ that there exists $c = c(n,m,s) > 0$ with

$$\int_{G_h(n,m)} d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s} \, d\mu_{n,m} V \le c d_H(p,q)^{-s}$$
$$\text{for all } (p,q) \in A \times A,\ p \ne q. \tag{4.1}$$

Although the idea is the same for all dimensions, the proof of (4.1) is more subtle for $n > 1$ than for $n = 1$ and requires careful work with the measure $\mu_{n,m}$. As preparation, we provide explicit formulas for the Heisenberg distance of points and the distance of the respective image points under vertical projection. Given points $p = (z, t)$ and $q = (\zeta, \tau)$, we observe that

$$d_H(p, q) = \sqrt[4]{|z-\zeta|^4 + (t-\tau-2\omega(\zeta, z))^2}$$

while $d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))$ is equal to

$$\sqrt[4]{|\pi_{V^\perp}(z-\zeta)|^4 + (t-\tau-2\omega(\pi_{V^\perp}(z), \pi_V(z)) + 2\omega(\pi_{V^\perp}(\zeta), \pi_V(\zeta)) - 2\omega(\pi_{V^\perp}(\zeta), \pi_{V^\perp}(z)))^2}.$$

Proof of Theorem 1.3. Let $0 < s < \dim_H A \leq 1$ and let $\mu \in \mathcal{M}(A)$ be a measure with $I_s(\mu) < \infty$. The existence of such μ is guaranteed by Theorem 3.2.

Let us assume that the inequality in (4.1) has been established. Then, for the Frostman measure $\mu \in \mathcal{M}(A)$, we find

$$\begin{aligned}
&\int_{G_h(n,m)} I_s((P_{\mathbb{V}^\perp})_\sharp \mu)\, \mathrm{d}\mu_{n,m} V \\
&\quad = \int_A \int_A \int_{G_h(n,m)} d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s}\, \mathrm{d}\mu_{n,m} V\, \mathrm{d}\mu p\, \mathrm{d}\mu q \\
&\quad \leq c \int_A \int_A d_H(p, q)^{-s}\, \mathrm{d}\mu p\, \mathrm{d}\mu q = c I_s(\mu) < \infty,
\end{aligned}$$

which proves that $I_s((P_{\mathbb{V}^\perp})_\sharp \mu)$ is finite for $\mu_{n,m}$ a.e. $V \in G_h(n, m)$. This in turn yields $\dim_H P_{\mathbb{V}^\perp}(A) \geq s$ for $\mu_{n,m}$ a.e. $V \in G_h(n, m)$, see Theorem 3.2. Taking the limit as s approaches $\dim_H A$ concludes the proof of Theorem 1.3. □

It remains to establish (4.1). The remainder of this section is devoted to this task.

To this end, we split the set $\{(p, q) \in A \times A : p \neq q\}$ into two subsets where either the first or the second summand in the formula for $d_H(p, q)$ is dominating. Restricting to these subsets makes the desired integral estimate easier to obtain.

More precisely, let us define

$$A_1 := \{(p, q) \in A \times A : |z-\zeta|^4 \geq (t-\tau-2\omega(\zeta, z))^2 \text{ with } p \neq q\}$$

and

$$A_2 := \{(p, q) \in A \times A : |z-\zeta|^4 < (t-\tau-2\omega(\zeta, z))^2\}.$$

Consider first points $p = (z, t)$ and $q = (\zeta, \tau)$ with $(p, q) \in A_1$. Observe that

$$d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s} \leq |\pi_{V^\perp}(z-\zeta)|^{-s},$$

and $d_H(p, q)^{-s} \geq 2^{-\frac{s}{4}} |z-\zeta|^{-s}$ for all $V \in G_h(n, m)$. Thus

$$\int_{G_h(n,m)} d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s}\, \mathrm{d}\mu_{n,m} V \leq \int_{G_h(n,m)} |\pi_{V^\perp}(z-\zeta)|^{-s}\, \mathrm{d}\mu_{n,m} V.$$

Similarly as in the proof of Theorem 1.2, we can apply Lemma 2.4 in order to show for $0 < s < 1$ (even for $0 < s < 2n - m$) that there is a constant $c = c(m, n, s)$ such that for $z - \zeta \in \mathbb{R}^{2n} \setminus \{0\}$, we have

$$\int_{G_h(n,m)} |\pi_{V^\perp}(z-\zeta)|^{-s}\, \mathrm{d}\mu_{n,m}V \le c|z-\zeta|^{-s}.$$

Notice that we may assume $z \neq \zeta$: indeed, if $z=\zeta$ and $|z-\zeta|^4 \ge (t-\tau-2\omega(z,\zeta))^2$ then it follows that $t=\tau$ and thus $p=q$. It follows that for $(p,q)\in A_1$ we have

$$\begin{aligned}\int_{G_h(n,m)} d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s}\, \mathrm{d}\mu_{n,m}V &\le \int_{G_h(n,m)} |\pi_{V^\perp}(z-\zeta)|^{-s}\, \mathrm{d}\mu_{n,m}V \\ &\le c|z-\zeta|^{-s} \lesssim d_H(p,q)^{-s},\end{aligned}$$

as desired.

Consider now points $p=(z,t)$ and $q=(\zeta,\tau)$ with $(p,q)\in A_2$; this is the more difficult case. Using the facts $\pi_{V^\perp}(z)=z-\pi_V(z)$ and $\omega(\pi_V(z),\pi_V(\zeta))=0$, we see that

$$\begin{aligned}&\omega(\pi_{V^\perp}(z),\pi_V(z)) - \omega(\pi_{V^\perp}(\zeta),\pi_V(\zeta)) + \omega(\pi_{V^\perp}(\zeta),\pi_{V^\perp}(z)) \\ &\quad = \omega(z-\zeta, \pi_V(z+\zeta)) + \omega(\zeta,z)\end{aligned}$$

for arbitrary $V\in G_h(n,m)$. Hence

$$I := \int_{G_h(n,m)} d_H(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(q))^{-s}\, \mathrm{d}\mu_{n,m}V$$

is bounded above by

$$\int_{G_h(n,m)} |t-\tau-2\omega(\zeta,z)-2\omega(z-\zeta,\pi_V(z+\zeta))|^{-s/2}\, \mathrm{d}\mu_{n,m}V. \tag{4.2}$$

If $z=\zeta$, then the integrand in (4.2) is equal to $|t-\tau|^{-s/2}$ and so $I=|t-\tau|^{-s/2}=d_H(p,q)^{-s}$. If $z=-\zeta$, then again $I=|t-\tau|^{-s/2}$. Moreover, in this case $d_H(p,q)=\sqrt[4]{16|z|^4+(t-\tau)^2} \le 2^{1/4}|t-\tau|^{1/2}$ (since $(p,q)\in A_2$) and so $I \le 2^{s/4}d_H(p,q)^{-s}$. In view of these facts we may assume $z\neq \pm\zeta$. We rewrite (4.2) in the form

$$\begin{aligned}&(|z+\zeta||z-\zeta|)^{-s/2}\int_{G_h(n,m)} \left|\frac{t-\tau-2\omega(\zeta,z)}{|z+\zeta||z-\zeta|}\right. \\ &\qquad \left. -2\omega\left(\frac{z-\zeta}{|z-\zeta|}, \pi_V\left(\frac{z+\zeta}{|z+\zeta|}\right)\right)\right|^{-s/2}\, \mathrm{d}\mu_{n,m}V.\end{aligned}$$

Our goal is to prove (4.1), which means that we should find a constant $c>0$ such that $I\le c d_H(p,q)^{-s}$ for all $(p,q)\in A_2$. This boils down to an estimate for an integral of the type $\int_{G_h(n,m)} |a-2\omega(v,\pi_V(w))|^{-s/2}\, \mathrm{d}\mu_{n,m}V$ for $a\in\mathbb{R}$ and $v,w\in\mathbb{S}^{2n-1}$. Such an estimate is contained in the following proposition.

Proposition 4.1. *Let* $0<s<1$. *The estimate*

$$\int_{G_h(n,m)} |a-2\omega(v,\pi_V(w))|^{-s/2}\, \mathrm{d}\mu_{n,m}V \lesssim 1 \tag{4.3}$$

holds for all $a\in\mathbb{R}$ *and* $v,w\in\mathbb{S}^{2n-1}$.

Assuming the validity of this proposition, let us complete the proof of the theorem. We consider two cases. Notice that $|2\omega(v,\pi_V(w))| \le 2$ for all $v, w \in \mathbb{S}^{2n-1}$. If

$$|a| = \frac{|t-\tau-2\omega(\zeta,z)|}{|z+\zeta||z-\zeta|} \ge 4,$$

then

$$I \lesssim (|z+\zeta||z-\zeta|)^{-s/2}\left(\frac{|t-\tau-2\omega(\zeta,z)|}{|z+\zeta||z-\zeta|}\right)^{-s/2} \lesssim d_H(p,q)^{-s}.$$

If

$$|a| = \frac{|t-\tau-2\omega(\zeta,z)|}{|z+\zeta||z-\zeta|} < 4,$$

then the result follows by an application of Proposition 4.1:

$$I \lesssim (|z+\zeta||z-\zeta|)^{-s/2} \lesssim |t-\tau-2\omega(\zeta,z)|^{-s/2} \lesssim d_H(p,q)^{-s}.$$

It remains to prove Proposition 4.1. We will divide the proof into two cases: the Lagrangian case ($m = n$) and the sub-Lagrangian case ($1 \le m < n$). The proof in these two cases will proceed by rather different methods.

Proof of Proposition 4.1 in the sub-Lagrangian case $m < n$. For $W \in G_h(n, m+1)$, let

$$G(W,m) = \{V \in G(2n,m) : V \subset W\}.$$

Then $G(W,m) \subset G_h(n,m)$. Let $\mu_{W,m}$ be the natural measure on $G(W,m)$. For a nonnegative Borel function f on $G_h(n,m)$, we have

$$\int f\, d\mu_{n,m} = \int_{G_h(n,m+1)} \int_{G(W,m)} f(V)\, \mathrm{d}\mu_{W,m} V\, \mathrm{d}\mu_{n,m+1} W. \tag{4.4}$$

Indeed, both sides define positive linear functionals on the space of nonnegative Borel functions on $G_h(n,m)$ and thus define Radon measures by Riesz's representation theorem. Moreover, both measures have total mass one and are invariant under the transitive action of $U(n)$. Since such measures are uniquely defined (see for instance [8, 2.7.11(2)]), the identity (4.4) follows. Writing $S(W) = \mathbb{S}^{2n-1} \cap W$, we obtain

$$\int f\, d\mu_{n,m} = c \int_{G_h(n,m+1)} \int_{S(W)} \tilde{f}(e)\, \mathrm{d}\mathcal{H}^m e\, \mathrm{d}\mu_{n,m+1} W,$$

where $\tilde{f}(e) = f(e^{\perp} \cap W)$.

Suppose now that $V \in G(W,m)$ for some $W \in G_h(n, m+1)$ with $m < n$. Then

$$\omega(v,\pi_V(w)) = \langle \mathbf{J}v, \pi_V(w)\rangle = \langle \pi_W(\mathbf{J}v), \pi_V(\pi_W(w))\rangle.$$

We may write $V = e^{\perp} \cap W$ for some $e \in S(W)$. Then $\pi_V(x) = x - \langle e,x\rangle e$ for $x \in W$ and

$$\omega(v,\pi_V(w)) = \langle \pi_W(\mathbf{J}v), \pi_W(w)\rangle - \langle \pi_W(w), e\rangle\langle \pi_W(\mathbf{J}v), e\rangle.$$

For technical reasons, we consider the set

$$(G_h(n,m+1))_{v,w} := \{W \in G_h(n,m) :\ \pi_W(\mathbf{J}v) \ne 0 \text{ and } \pi_W(w) \ne 0\}.$$

By Lemma 2.4 we have $\mu_{n,m}(G_h(n,m) \setminus (G_h(n,m))_{v,w}) = 0$. We may write the integral in (4.3) as

$$\begin{aligned}
&\int |a - 2\omega(v, \pi_V(w))|^{-s/2}\, \mathrm{d}\mu_{n,m} V \\
&\quad = c \int_{(G_h(n,m+1))_{v,w}} \int_{S(W)} |a - 2\langle \pi_W(\mathbf{J}v), \pi_W(w)\rangle \\
&\qquad + 2\langle \pi_W(w), e\rangle \langle \pi_W(\mathbf{J}v), e\rangle|^{-s/2} \mathrm{d}\mathcal{H}^m e\, \mathrm{d}\mu_{n,m+1} W \\
&\quad = c \int_{(G_h(n,m+1))_{v,w}} |\pi_W(\mathbf{J}v)|^{-s/2} |\pi_W(w)|^{-s/2} \int_{S(W)} |b(W) \\
&\qquad + 2\langle \bar{v}, e\rangle \langle \bar{w}, e\rangle|^{-s/2}\, \mathrm{d}\mathcal{H}^m e\, \mathrm{d}\mu_{n,m+1} W
\end{aligned} \tag{4.5}$$

where

$$b(W) = \frac{a - 2\langle \pi_W(\mathbf{J}v), \pi_W(w)\rangle}{|\pi_W(\mathbf{J}v)|\,|\pi_W(w)|},$$

$\bar{v} = \pi_W(\mathbf{J}v)/|\pi_W(\mathbf{J}v)|$, and $\bar{w} = \pi_W(w)/|\pi_W(w)|$.

We shall prove that the inner integral is uniformly bounded.

Lemma 4.2. *Suppose* $0 < s < 1$ *and* $m \geq 1$. *Then for* $v, w \in \mathbb{S}^m$ *and* $b \in \mathbb{R}$, *we have*

$$\int_{\mathbb{S}^m} |b + 2\langle v, e\rangle \langle w, e\rangle|^{-s/2}\, \mathrm{d}\mathcal{H}^m e \lesssim 1. \tag{4.6}$$

Assuming the lemma, we establish (4.3). We may identify $S(W)$ with $\mathbb{S}^m$. In (4.5) the inner integral is uniformly bounded for all $\overline{v}$, $\overline{w}$ and W. Hence

$$\begin{aligned}
&\int |a - 2\omega(v, \pi_V(w))|^{-s/2}\, \mathrm{d}\mu_{n,m} V \\
&\quad \lesssim \int_{(G_h(n,m+1))_{v,w}} |\pi_W(\mathbf{J}v)|^{-s/2} |\pi_W(w)|^{-s/2}\, \mathrm{d}\mu_{n,m+1} W \\
&\quad = \int_{G_h(n,m+1)} |\pi_W(\mathbf{J}v)|^{-s/2} |\pi_W(w)|^{-s/2}\, \mathrm{d}\mu_{n,m+1} W \\
&\quad \leq \left(\int_{G_h(n,m+1)} |\pi_W(\mathbf{J}v)|^{-s}\, \mathrm{d}\mu_{n,m+1} W \right)^{1/2} \left(\int_{G_h(n,m+1)} |\pi_W(w)|^{-s}\, \mathrm{d}\mu_{n,m+1} W \right)^{1/2} \\
&\quad \lesssim 1
\end{aligned}$$

by Corollary 3.12 in [16] or as in the proof of Theorem 1.2 above. This concludes the proof of Proposition 4.1 in the sub-Lagrangian case, except for the proof of Lemma 4.2. □

The integrability of polynomials to negative powers is well-studied in complex algebraic geometry where it is related to the log canonical threshold: for an informative account see [18]. But such results cannot be easily applied to integrals over manifolds such as spheres or Grassmannians. In our explicit context we prefer to prove Lemma 4.2 directly.

Proof of Lemma 4.2. First we observe that it suffices to prove the statement for b lying in a compact interval centered at the origin, since the integrand is clearly uniformly bounded for large $|b|$. Define $f(e) = b + 2\langle v, e\rangle \langle w, e\rangle$. It suffices to show that every point $e \in \mathbb{S}^m$, $f(e) \neq 0$

or some of the first or second order partial derivatives of f along $\mathbb{S}^m$ at e does not vanish. If this is true then, by continuity and compactness, there is $c > 0$ independent of b, v and w such that one of these quantities is at least c at every $e \in \mathbb{S}^m$. This allows us to deduce (4.6) from sub-level set estimates as in Lemma 3.4 and its proof in [5].[1]

Fix $e \in \mathbb{S}^m$, and for $u \in \mathbb{S}^m \cap e^\perp$ and $\xi \in [0,1]$ define

$$f_u(\xi) = f\left(\sqrt{1-\xi^2}e + \xi u\right) = b + 2\left\langle v, \sqrt{1-\xi^2}e + \xi u\right\rangle\left\langle w, \sqrt{1-\xi^2}e + \xi u\right\rangle.$$

Direct computations give $f_u'(0) = 2(\langle v,e\rangle\langle w,u\rangle + \langle v,u\rangle\langle w,e\rangle)$ and $f_u''(0) = 4(\langle v,u\rangle\langle w,u\rangle - \langle v,e\rangle\langle w,e\rangle)$. Consequently,

$$\frac{1}{4}f_u'(0)^2 + \frac{1}{16}f_u''(0)^2 = \left(\langle v,e\rangle^2 + \langle v,u\rangle^2\right)\left(\langle w,e\rangle^2 + \langle w,u\rangle^2\right). \tag{4.7}$$

We will show that there exists $u \in \mathbb{S}^m \cap e^\perp$ such that either $f_u'(0) \neq 0$ or $f_u''(0) \neq 0$. Suppose not. Then at least one of the two factors on the right hand side of (4.7) is equal to zero for some choice of u. Without loss of generality assume that it is the first factor. Then $\langle v,e\rangle = 0$, so $v \in e^\perp$ and the desired conclusion holds unless $0 = f_v'(0) = f_v''(0)$, i.e., unless $\langle w,e\rangle = \langle w,v\rangle = 0$. In the latter case, we also have $w \in e^\perp$. Then $\alpha v + \beta w \in \mathbb{S}^m \cap e^\perp$ for all α and β satisfying $\alpha^2 + \beta^2 = 1$. If $\langle v,w\rangle = 0$ then choose any such pair (α,β) with $\alpha\beta \neq 0$, while if $\langle v,w\rangle \neq 0$ then choose $\alpha = (1+\langle v,w\rangle)^{-1/2}$ and $\beta = \langle v,w\rangle\,(1+\langle v,w\rangle)^{-1/2}$. With these choices of α and β we compute $f''_{\alpha v+\beta w}(0) = 4\big(\alpha\beta(1+\langle v,w\rangle^2) + \langle v,w\rangle\big) = 8\langle v,w\rangle \neq 0$. □

We now give another argument which covers the Lagrangian case $m = n$ in the statement of Proposition 4.1.

Proof of Proposition 4.1 in the Lagrangian case $m = n$. Let

$$V_0 = \mathbb{R}^n \times \{0\} = \operatorname{span}\{e_1, \ldots, e_n\}.$$

According to Proposition 2.7, the Lagrangian Grassmannian $G_h(n,n)$ is identified with the homogeneous space $U(n)/O(n)$. By the definition of the measure $\mu_{n,m}$ (see (2.1)) we have

$$\int_{U(n)} |a - 2\omega(v, \pi_{gV_0}(w))|^{-s/2}\, d\vartheta_n(g) = \int_{G_h(n,n)} |a - 2\omega(v, \pi_V(w))|^{-s/2}\, d\mu_{n,n}(V).$$

Consequently, it suffices to consider the integral

$$\int_{U(n)} |a - 2\omega(v, \pi_{gV_0}(w))|^{-s/2}\, d\vartheta_n(g) \tag{4.8}$$

which we will do from now on.

The standard basis $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V_0. For every $g \in U(n)$, the family $\{ge_1, \ldots, ge_n\}$ is an orthonormal basis for gV_0. Thus $\pi_{gV_0}(w) = \sum_{j=1}^n \langle w, ge_j\rangle\, ge_j$ and

$$\omega(v, \pi_{gV_0}(w)) = \langle \mathbf{J}v, \pi_{gV_0}(w)\rangle = \sum_{j=1}^n \langle w, ge_j\rangle\,\langle \mathbf{J}v, ge_j\rangle = \sum_{j=1}^n \langle g^T w, e_j\rangle\,\langle g^T\mathbf{J}v, e_j\rangle.$$

[1] Lemma 3.4 in [5] is formulated in $\mathbb{R}^n$, but easily transfers to $\mathbb{S}^m$ via chart maps.

Since g is symplectic, g^T commutes with $\mathbf{J}$ and so

$$\omega(v, \pi_{gV_0}(w)) = \sum_{j=1}^{n} \langle g^T w, e_j\rangle \langle \mathbf{J} g^T v, e_j\rangle.$$

Our goal now is to study the behavior of the function $F : U(n) \to \mathbb{R}$ given by

$$F(g) := a - 2\omega(v, \pi_{gV_0}(w)) = a - 2\sum_{j=1}^{n} \langle g^T w, e_j\rangle \langle \mathbf{J} g^T v, e_j\rangle \tag{4.9}$$

in a neighborhood of the hypersurface $Z = F^{-1}(0)$. As in the previous proof, by an appeal to the sub-level set estimates in [5, Lemma 3.4], it suffices to check that F vanishes to no worse than second order along Z. Again, to make the argument precise we should work in coordinate charts. To simplify the exposition we show here that at each $g \in Z$, either $DF(g) \neq 0$ or $D^2F(g) \neq 0$. To this end, we will exhibit explicit one-parameter families (g_s) passing through each $g \in Z$ so that, setting $f(s) = F(g_s)$, we have either $f'(0) \neq 0$ or $f''(0) \neq 0$.

At this stage to further simplify the exposition we switch to the complex unitary group $U(n, \mathbb{C}) = \{g \in M(n, \mathbb{C}) : g^* g = \mathbf{I}_n\}$. Recall that $U(n, \mathbb{C})$ maps onto $U(n)$ by the monomorphism

$$g = \mathbf{A} + \mathbf{i}\mathbf{B} \mapsto \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}.$$

Identifying the vectors $v, w \in \mathbb{S}^{2n-1}$ as vectors in the unit sphere of $\mathbb{C}^n$, we observe that the function F given in (4.9) takes the form

$$F(g) = a + 2\sum_{j=1}^{n} \operatorname{Re}(g^* w \bullet e_j) \operatorname{Im}(g^* v \bullet e_j).$$

Here we wrote $z \bullet w$ for the standard Hermitian inner product on $\mathbb{C}^n$.

The tangent space to $U(n, \mathbb{C})$ at g can be identified as follows:

$$T_g U(n, \mathbb{C}) = \{\mathbf{B} \in M(n, \mathbb{C}) : g^* \mathbf{B} + \mathbf{B}^* g = 0\}.$$

It is clear that $T_g U(n, \mathbb{C}) = g T_{\mathbf{I}_n} U(n, \mathbb{C})$ where $T_{\mathbf{I}_n} U(n, \mathbb{C}) = \mathfrak{u}(n, \mathbb{C}) = \{\mathbf{A} \in M(n, \mathbb{C}) : \mathbf{A} + \mathbf{A}^* = 0\}$ denotes the Lie algebra of $U(n, \mathbb{C})$ consisting of skew-Hermitian matrices.

For fixed $\mathbf{A} \in \mathfrak{u}(n, \mathbb{C})$ consider the one-parameter family (g_s) given by $g_s = g \exp(s\mathbf{A})$. Then $g_0 = g$ and $\frac{d}{ds} g_s|_{s=0} = g\mathbf{A}$.

Lemma 4.3. *Let*

$$\mathbf{A} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & 0 \end{pmatrix},$$

let (g_s) be the one-parameter family described above, and let $f(s) = F(g_s)$. Then $f'(0) = -2 \operatorname{Re}(\hat{v}_1 \hat{w}_1)$ and $f''(0) = -4 \operatorname{Im}(\hat{v}_1 \hat{w}_1)$.

Here and henceforth we denote by $\hat{v}$, resp. $\hat{w}$, the vector $g^* v$, resp. $g^* w$. Note that $\hat{v}$ and $\hat{w}$ are still unit vectors, since g is unitary.

Remark 4.4. The index 1 in Lemma 4.3 can be replaced by any index k between 1 and n.

Corollary 4.5. *If $f'(0) = f''(0) = 0$ for all such one-parameter families (for each $k = 1, \dots, n$), then for each k either $\hat{v}_k = 0$ or $\hat{w}_k = 0$.*

Proof of Lemma 4.3. With $\mathbf{A}$ given as in the statement of the lemma, we compute

$$g_s = g\,\exp(s\mathbf{A}) = g\begin{pmatrix} e^{\mathbf{i}s} & 0 \\ 0 & \mathbf{I}_{n-1} \end{pmatrix}$$

and $f(s) = F(g_s) = a + 2\operatorname{Re}(e^{-\mathbf{i}s}\hat{w}_1)\operatorname{Im}(e^{-\mathbf{i}s}\hat{v}_1) + 2\sum_{j=2}^{n}\operatorname{Re}(\hat{w}_j)\operatorname{Im}(\hat{v}_j)$. Direct computation yields the stated values for $f'(0)$ and $f''(0)$. $\square$

Lemma 4.6. *Let*

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{i} & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

let (g_s) be the one-parameter family described above, and let $f(s) = F(g_s)$. Then $f'(0) = -2\operatorname{Re}(\hat{v}_1\,\hat{w}_2 + \hat{v}_2\,\hat{w}_1)$ and $f''(0) = -4\operatorname{Im}(\hat{v}_1\,\hat{w}_2 + \hat{v}_2\,\hat{w}_1)$.

Remark 4.7. The indices 1 and 2 in Lemma 4.6 can be replaced by any distinct indices k and l between 1 and n.

Corollary 4.8. *If $f'(0) = f''(0) = 0$ for all such one-parameter families (for each pair $k, l = 1, \dots, n$, $k \neq l$), then for each $k \neq l$ we have $\hat{v}_k\,\hat{w}_l + \hat{v}_l\,\hat{w}_k = 0$.*

The proof of Lemma 4.6 proceeds along similar lines as that of Lemma 4.3.

Using these two lemmas, let us complete the proof of Proposition 4.1. Suppose that $f'(0) = f''(0) = 0$ for all the one-parameter families described in Lemmas 4.3 and 4.6, for some choice of $\hat{v}$ and $\hat{w}$ in $\mathbb{S}^{2n-1}$. According to Corollary 4.5, either $\hat{v}_1$ or $\hat{w}_1$ is equal to zero: without loss of generality assume $\hat{v}_1 = 0$. By Corollary 4.8, either $\hat{w}_1 = 0$ or $\hat{v}_2 = \hat{v}_3 = \dots = \hat{v}_n = 0$. Since all entries of $\hat{v}$ cannot be nonzero we must have $\hat{w}_1 = 0$. The same argument can be applied with the index 1 replaced by any index k to conclude that all of the entries of $\hat{v}$ and $\hat{w}$ are equal to zero. But this obviously contradicts the fact that $\hat{v}, \hat{w} \in \mathbb{S}^{2n-1}$. This contradiction implies that $(f'(0), f''(0)) \neq (0,0)$ for at least one of the one-parameter families described above. From here we see that either $DF(g)$ or $D^2F(g)$ is nonzero. As previously indicated, this fact suffices to finish the proof of Proposition 4.1 in the Lagrangian case. $\square$

Remark 4.9. An argument as above might also work in the case $m < n$ with some additional choices of one-parameter families (g_s). In fact, we can proceed as above to define F with summation only up to m. Then, with the previous paths (g_s), if $f(0) = f'(0) = f''(0) = 0$, we can conclude that $a = 0$. So it would be enough to find a separate argument in the case $a = 0$. Note that even though $a \neq 0$ in the case of pairs (p, q) in A_2 we cannot use this fact, since in verifying Proposition 4.1 we referred to compactness.

On the other hand, for the case $m = n$ there could also be a way of using a method similar to that which we used for $m < n$ by an appropriate splitting of the space of Lagrangian subspaces. We have not pursued seriously either of these alternatives since we thought that presenting two somewhat different methods might be useful in other occasions.

5. Universal dimension estimates for vertical projections

The goal of this section is to prove Theorem 1.4.

First, we prove the upper bound on t in (1.6). The estimate $t \le 2s$ follows from the local $\frac{1}{2}$-Hölder continuity of $P_{\mathbb{W}}$. To establish the estimate $t \le s+1$ we compute

$$\dim_H P_{\mathbb{W}}(A) \le \dim_E P_{\mathbb{W}}(A) + 1 \le \dim_E A + 1 \le \dim_H A + 1.$$

Here the first and last inequalities follow from the dimension comparison principle in $\mathbb{H}^n$, while the second inequality follows from the fact that the (nonlinear) mapping $P_{\mathbb{W}}$ acting on $\mathbb{R}^{2n+1}$ is locally Lipschitz and hence reduces Hausdorff dimension.

The inequality $t \le 2n+2-m$ is obvious since $2n+2-m$ is the Heisenberg dimension of the ambient space $\mathbb{W}$ containing the set $P_{\mathbb{W}}(A)$. It remains to establish the inequality

$$t \le \frac{1}{2}(s-m) + n + 1. \tag{5.1}$$

We use a covering argument similar to that used in the proof of Proposition 4.3 in [2]. The projection $P_{\mathbb{W}}(B_H(p,r))$ looks like the Cr^2-vertical thickening of a $(2n-m)$-dimensional algebraic set Z transverse to the vertical direction.

In the following lemma, we denote by $H_p = \{p * (z,0) : z \in \mathbb{R}^{2n}\}$ the maximal horizontal affine subspace of $\mathbb{H}^n$ passing through the point p.

Lemma 5.1. *Denoting $Z := P_{\mathbb{W}}\left(B_H(p,r) \cap H_p\right)$, we have $P_{\mathbb{W}}(B_H(p,r)) \subseteq N_E(Z, Cr^2)$, where $N_E(Z, Cr^2)$ denotes the Euclidean Cr^2 neighborhood of Z in $\mathbb{W}$.*

The proof of this lemma is similar to that found in [2].

Lemma 5.2. *The set $N_E(Z, Cr^2)$ can be covered by Heisenberg balls $B_H(p_j, r^2) \cap \mathbb{W}$, $p_j \in \mathbb{W}$, $1 \le j \le N$, where*

$$N = O\left(\left(\frac{1}{r^2}\right)\left(\frac{1}{r}\right)^{2n-m}\right) = O\left(\frac{1}{r^{2n+2-m}}\right). \tag{5.2}$$

Proof of Lemma 5.2. First we estimate the number of Euclidean balls of radius r^2 needed to cover $N_E(Z, Cr^2)$. Since the Euclidean diameter of Z is $\lesssim r$, we need for each horizontal direction $O(\frac{1}{r})$ Euclidean balls of radius r^2. As $\mathbb{W}$ is spanned by $2n-m$ horizontal and one vertical direction, this amounts to $O(\frac{1}{r^{2n-m}})$ Euclidean balls of radius r^2 needed to cover $N_E(Z, Cr^2)$.

Next, we verify how many Heisenberg balls of radius r^2 (intersected with $\mathbb{W}$) are needed to cover one Euclidean ball of radius r^2 (intersected with $\mathbb{W}$). As $B_H(p_j, r^2)$ looks roughly like a parallelepiped with horizontal edges of length $O(r^2)$ and vertical height $O(r^4)$ for small r, we need $O(\frac{1}{r^2})$ Heisenberg balls of radius r^2 to cover such a ball. (Compare Lemma 6.4.)

Taken together, these two estimates yield the quantity in (5.2). □

We now prove the indicated dimension estimate (5.1). Fix $s' > s$ and $\epsilon > 0$ and cover A with balls $B_H(p_i, r_i)$ so that $r_i < \epsilon$ and $\sum_i r_i^{s'} < \epsilon$. Then $P_{\mathbb{W}}(A)$ is covered by the sets $P_{\mathbb{W}}(B_H(p_i, r_i))$. By Lemma 5.2, the set $P_{\mathbb{W}}(A)$ can be covered by balls $B_H(p_{ij}, r_i^2) \cap \mathbb{W}$, where

$1 \le j \le N_i = O(r_i^{m-2n-2})$. Let $t' = \frac{1}{2}(s'-m)+n+1$. Then

$$\mathcal{H}^{t'}_{H,\epsilon^2}(P_{\mathbb{W}}(A)) \lesssim \sum_i \sum_j (r_i)^{2t'} = \sum_i N_i r_i^{2t'} \le C \sum_i r_i^{2t'+m-2n-2} = C \sum_i r_i^{s'} < C\epsilon.$$

Letting $\epsilon \to 0$ gives $\mathcal{H}^{t'}_H(P_{\mathbb{W}}(A)) = 0$. Letting $s' \to s$ (so $t' \to t$) we conclude that $\dim_H P_{\mathbb{W}}(A) \le \frac{1}{2}(\dim_H A - m) + n + 1$ as desired. This completes the proof of the upper bound in (1.6).

Next, we turn to the lower bound in (1.6). We begin with the lower bound $t \ge \frac{1}{2}(s-m)$. The argument is similar to that used in the proof of Proposition 4.7 in [2]. Our starting point is the following sequence of lemmas, cf. Lemma 4.8 and related statements in the proof of Proposition 4.7 in [2].

Lemma 5.3. *There exists $C > 0$ so that $\|a^{-1} * b * a\|_H^4 \le \|b\|_H^4 + C\|b\|_H^2$ whenever $a, b \in \mathbb{H}^n$ satisfy $\|a\|_H \le 1$ and $\|b\|_H \le 1$.*

Lemma 5.4. *There exists $C > 0$ so that $P_{\mathbb{W}}^{-1}(B_H(q,r) \cap \mathbb{W}) \cap B \subset N_H(q * \mathbb{V}, C\sqrt{r})$ whenever $q \in \mathbb{W}$ and $0 < r \le 1$.*

Here

$$B = B_H(0,1) \tag{5.3}$$

denotes the unit ball in the metric d_H on $\mathbb{H}^n$, while $N_H(A,\epsilon)$ denotes the ϵ-neighborhood of a set A in $(\mathbb{H}^n, d_H)$.

Lemma 5.5. *There exists an absolute constant $C > 0$ and $N = O(r^{-m/2})$ points p_j contained in the fiber $P_{\mathbb{W}}^{-1}(q)$ so that the family $\{B_H(p_j, C\sqrt{r}) : j = 1, \ldots, N\}$ covers $P_{\mathbb{W}}^{-1}(B_H(q,r)) \cap B$.*

Now let $A \subset \mathbb{H}^n$ be a Borel set; we claim that $\dim_H P_{\mathbb{W}}(A) \ge \frac{1}{2}(\dim_H A - m)$. Without loss of generality we may assume that A is bounded, and (after a dilation) we may assume that $A \subset B$ where B is as in (5.3).

Fix now $t' > t = \dim_H P_{\mathbb{W}}(A)$ and $\epsilon > 0$ and let $\{B_H(q_i, r_i)\}$ be a family of balls with $q_i \in \mathbb{W}$ so that $P_{\mathbb{W}}(A) \subset \bigcup_i B_H(q_i, r_i)$ and $\sum_i r_i^{t'} < \epsilon$. We can apply Lemma 5.5 to each ball $B_H(q_i, r_i)$, obtaining balls $\{B_H(p_{ij}, C\sqrt{r_i}) : 1 \le j \le N_i\}$ with $N_i = O(r_i^{-m/2})$ so that $A \subset \bigcup_{i,j} B_H(p_{ij}, C\sqrt{r_i})$. Then, with $s' = 2t' + m$ we find

$$\begin{aligned}\sum_{i,j} (\operatorname{diam} B_H(p_{ij}, C\sqrt{r_i}))^{s'} &\le C \sum_{i,j} r_i^{s'/2} \le C \sum_i N_i r_i^{s'/2} \\ &\le C \sum_i r_i^{(s'-m)/2} = C \sum_i r_i^{t'} < C\epsilon.\end{aligned}$$

Since ϵ can be made arbitrarily small, we conclude that $s = \dim_H A \le s' = 2t' + m$. Letting $t' \to t$ we conclude that $s \le 2t + m$ as desired.

We now prove the remaining dimension lower bound $t \ge \max\{s-m-1, 2s-2n-m-2\}$. The argument for this case is similar to that used in the proof of Proposition 4.9 in [2] and uses the Dimension Comparison Principle for $\mathbb{H}^n$ together with a slicing theorem. However, in contrast with the case considered in [2], we use here the Heisenberg slicing Theorem 1.5 instead of the Euclidean slicing theorem. (We will prove Theorem 1.5 in the following section.)

Let $A \subset \mathbb{H}^n$ be a Borel set and let $V \in G_h(n,m)$, and recall that our goal is to prove $t \geq \max\{s-m-1, 2s-2n-m-2\}$ where $s = \dim_H A$ and $t = \dim_H P_{\mathbb{V}^\perp}(A)$. We may assume without loss of generality that $s > m+2$ (otherwise, the preceding lower bound $t \geq (s-m)/2$ is stronger). This estimate will follow from the Dimension Comparison Principle and the Euclidean dimension estimate

$$\dim_E P_{\mathbb{V}^\perp}(A) \geq \dim_H A - m - 1. \tag{5.4}$$

In order to prove (5.4), we make use of the fact that for a given $V \in G_h(n,m)$, the set $\{U \in G_h(n,m) : U^\perp \cap V = \{0\}\}$ is a non-empty open subset of $G_h(n,m)$ and therefore of positive $\mu_{n,m}$ measure. This observation, together with Theorem 1.5, allows us to choose for small $\varepsilon > 0$ an element $U \in G_h(n,m)$ and $u \in U$ such that

1. $\pi_{V^\perp}|_{U^\perp} : U^\perp \to V^\perp$ is one-to-one, and
2. $\dim_H(A \cap \mathbb{U}_u^\perp) \geq \dim_H A - m - \varepsilon$.

For such a choice, the map $P_{\mathbb{V}^\perp}|_{\mathbb{U}_u^\perp} : \mathbb{U}_u^\perp \to \mathbb{V}^\perp$ is bijective. Surjectivity is obvious. To see that the map is injective, note that points in $\mathbb{U}_u^\perp$ can be written in a unique way as $(z_{U^\perp}, t) * (u, 0)$. Then

$$P_{\mathbb{V}^\perp}((z_{U^\perp}, t) * (u,0)) = P_{\mathbb{V}^\perp}((\zeta_{U^\perp}, \tau) * (u,0)),$$

or, equivalently,

$$\begin{aligned}&(\pi_{V^\perp}(z_{U^\perp} + u), t + 2\omega(z_{U^\perp}, u) - 2\omega(\pi_{V^\perp}(z_{U^\perp} + u), \pi_V(z_{U^\perp} + u)))\\ &\quad = (\pi_{V^\perp}(\zeta_{U^\perp} + u), \tau + 2\omega(\zeta_{U^\perp}, u) - 2\omega(\pi_{V^\perp}(\zeta_{U^\perp} + u), \pi_V(\zeta_{U^\perp} + u))).\end{aligned}$$

The latter equation implies that

$$\pi_{V^\perp}(z_{U^\perp} + u) = \pi_{V^\perp}(\zeta_{U^\perp} + u)$$

and thus

$$\pi_{V^\perp}(z_{U^\perp}) = \pi_{V^\perp}(\zeta_{U^\perp}).$$

Since $\pi_{V^\perp}|_{U^\perp}$ is one-to-one, it follows that $z_{U^\perp} = \zeta_{U^\perp}$ and then, by considering the second component, also that $t = \tau$. This proves that $P_{\mathbb{V}^\perp}|_{\mathbb{U}_u^\perp}$ is bijective. In fact, it is locally bi-Lipschitz with respect to the Euclidean metric d_E. Indeed, consider the inverse

$$(P_{\mathbb{V}^\perp}|_{\mathbb{U}_u^\perp})^{-1} : \mathbb{V}^\perp \to \mathbb{U}_u^\perp, \qquad (z', t') \mapsto (Z, T).$$

We will in the following give an explicit formula for this inverse map from which it can be seen that it is smooth, and thus local Lipschitz continuity with respect to the Euclidean metric follows. From

$$P_{\mathbb{V}^\perp}(Z, T) = P_{\mathbb{V}^\perp}(z_{U^\perp} + u, t + 2\omega(z_{U^\perp}, u)) \stackrel{!}{=} (z', t') \in \mathbb{V}^\perp,$$

it follows that

$$Z = (\pi_{V^\perp}|_{U^\perp})^{-1}(z' - \pi_{V^\perp}(u)) + u$$

and

$$T = t' + 2\omega(\pi_{V^\perp}(Z), \pi_V(Z))$$

$$= t' + 2\omega\Big(\pi_{V^\perp}\big((\pi_{V^\perp}|_{U^\perp})^{-1}(z' - \pi_{V^\perp}(u)) + u\big), \pi_V\big((\pi_{V^\perp}|_{U^\perp})^{-1} \times (z' - \pi_{V^\perp}(u)) + u\big)\Big)$$

Note that $\pi_{V^\perp}|_{U^\perp}$ is linear, so its inverse (which exists as a map from $V^\perp$ to $U^\perp$ by assumption) is also linear. It follows that the map

$$\mathbb{V}^\perp \to \mathbb{U}_u^\perp, \qquad (z', t') \mapsto (Z, T),$$

which is given by the above formulas, is smooth and thus locally Lipschitz.

The desired dimension bound (5.4) now follows from the Dimension Comparison Principle and our choice of U and u, as follows:

$$\begin{aligned}\dim_E P_{\mathbb{V}^\perp}(A) &\geq \dim_E P_{\mathbb{V}^\perp}(A \cap \mathbb{U}_u^\perp) = \dim_E A \cap \mathbb{U}_u^\perp \\ &\geq \dim_H A \cap \mathbb{U}_u^\perp - 1 = \dim_H A - m - \varepsilon - 1.\end{aligned}$$

Letting $\varepsilon \to 0$ gives (5.4).

The proof of the universal lower bound for vertical Heisenberg projections can now be completed using another application of the Dimension Comparison Principle, this time for vertical subgroups:

$$\begin{aligned}\dim_H P_{\mathbb{W}}(A) &\geq \max\{\dim_H A - m - 1, 2(\dim_H A - m - 1) - (2n - m)\} \\ &= \max\{s - m - 1, 2s - 2n - m - 2\}.\end{aligned}$$

6. A slicing theorem in the Heisenberg group

The goal of this section is to prove Theorem 1.5 which computes the Hausdorff dimension of the intersection of a set in the Heisenberg group with cosets of a generic vertical subgroup. We also prove related statements on the dimensions of slices by cosets of vertical subgroups.

Let $A \subset \mathbb{H}^n$ be a Borel set with positive and finite measure in its Hausdorff dimension. Our goal is to estimate the dimensions of the slices of A by cosets

$$\mathbb{V}_u^\perp := \mathbb{V}^\perp * u, \quad u \in \mathbb{V},$$

of a given vertical subgroup $\mathbb{V}^\perp$.

The universal upper bound $\dim_H(A \cap \mathbb{V}_u^\perp) \leq \dim_H A - m$ is easily established for all V in $G_h(n, m)$ and $\mathcal{H}^m$ almost every $u \in \mathbb{V}$. More difficult is the verification of the corresponding lower estimate:

$$\mathcal{H}^m\big(\{u \in \mathbb{V}:\ \dim_H(A \cap (\mathbb{V}^\perp * u)) \geq \dim_H A - m\}\big) > 0 \quad \text{for } \mu_{n,m} \text{ a.e. } V. \tag{6.1}$$

Here we will need to impose the restriction $\dim_H A > m + 2$ as postulated in the statement of Theorem 1.5.

We begin this section with several auxiliary results which will enable us to prove (6.1). The strategy is similar to that in the Euclidean case. First, we 'slice' a given Radon measure μ on $\mathbb{H}^n$ with the cosets $\mathbb{V}_u^\perp$ for $V \in G_h(n, m)$ and $u \in \mathbb{V}$. We obtain measures $\mu_{V,u}$ on $\mathbb{V}_u^\perp$, see (6.5). We discuss properties of these measures $\mu_{V,u}$; in particular we show that if μ satisfies a growth condition for $s > m+2$, then many $\mu_{V,u}$ have finite $(\sigma - m)$-energy, $s > \sigma > m+2$. This is the content of Proposition 6.7 below. It allows us to use the measures $\mu_{V,u}$ in order to estimate the dimension of $A \cap \mathbb{V}_u^\perp$ provided that $\mu_{V,u} \in \mathcal{M}(\mathbb{V}_u^\perp)$, in particular provided that $\mu_{V,u}$ is positive for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$ and a positive $\mathcal{H}^m$ measure set of $u \in \mathbb{V}$.

Our point of departure is the subsequent proposition.

Proposition 6.1. *Let $m \in \{1,\dots,n\}$ and $s > m+2$. Assume that μ is a Radon measure on $\mathbb{H}^n$ with compact support such that*

$$\mu(B_H(p,r)) \le r^s \quad \text{for all } p \in \mathbb{H}^n, r > 0. \tag{6.2}$$

Then

$$P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor \mathbb{V} \quad \text{for } \mu_{n,m} \text{ almost every } V \in G_h(n,m). \tag{6.3}$$

Before beginning the proof of this result, we record the following corollary, which completes the proof of Theorem 1.1.

Corollary 6.2. *Let $A \subset \mathbb{H}^n$ be a Borel set with $\dim_H A > m+2$. Then $\mathcal{H}^m(P_{\mathbb{V}}A) > 0$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$.*

Proof. Fix s with $m+2 < s < \dim_H A$. By Frostman's lemma, there exists a positive Radon measure μ supported on A satisfying $\mu(B_H(p,r)) \le r^s$ for all $p \in \mathbb{H}^n$ and $r > 0$. By Proposition 6.1, $P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m\lfloor\mathbb{V}$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$. Since $(P_{\mathbb{V}\sharp}\mu)(P_{\mathbb{V}}A) = \mu(A) > 0$ the conclusion follows. □

As in the Euclidean case (see, for instance [16, Theorem 9.7]), the proof of Proposition 6.1 uses differentiation theory. But in contrast to that case, it is not possible in this setting to bound the integral of the lower derivative by the energy integral. In order to remedy this, we have imposed the growth condition (6.2) on μ instead of requiring finiteness of the s-energy.

We will employ a covering argument by balls together with (6.2) to ensure that the given integral is bounded. To this end, we will need several covering results (Lemmas 6.3–6.6), similar to the one found in [3], but for higher-dimensional Heisenberg groups. Variants of Lemmas 6.3 and 6.4 can also be found in [4].

Lemma 6.3. *Let $A \subset \mathbb{R}^{2n+1}$ be bounded, let $r > 0$, let $p \in \mathbb{R}^{2n+1}$ and let $p' \in A$. Then, denoting by $*$ the group law in $\mathbb{H}^n$ and by $+$ the Euclidean group law in $\mathbb{R}^{2n+1}$, we have*

- *$p' * B_E(p,r) \subset B_E(p' * p, Cr)$ for some C depending only on $\operatorname{diam}_E A$, and*
- *if $p' * p + p'' \in p' * B_E(p,r)$, then $|\pi(p'')| \le r$.*

Here as before $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ denotes the projection $\pi(z,t) = z$. The proof of Lemma 6.3 is analogous to that in [3].

Lemma 6.4. *Let $A \subset \mathbb{R}^{2n+1}$ be bounded. Then there exists a constant C depending only on n and $\operatorname{diam}_E A$ so that any Euclidean ball $B_E(p,r)$ with $p \in A$ and $0 < r < 1$ can be covered by Heisenberg balls $B_H(p_1,r),\dots,B_H(p_M,r)$ with $M \le C/r$.*

The proof is essentially the same as that in [3]. Translate the center of the given Euclidean ball to the origin via Heisenberg left translation. The translated set can be covered by parallelepipeds whose side lengths are comparable to r, except for the 'height' (side length in t-direction) which is comparable to r^2. The number of parallelepipeds needed for such a covering can be estimated by Lemma 6.3. Since each parallelepiped can be contained in a ball of radius comparable to r, the result follows by applying another left translation and using the doubling property of the Heisenberg group.

From Lemma 6.4 we quickly deduce the following covering theorem.

Lemma 6.5. *Let $b \geq 0$ and $\rho_0 > 1$. Consider the set $A = \{z_0 \in \mathbb{R}^{2n} : |z_0| \leq b\}$. Then there exists $C(n, b, \rho_0) > 0$ such that for all $0 < r < 1$ and $z_0 \in A$ the set*

$$C_r(z_0) := \{(z, t) \in \mathbb{H}^n : |z - z_0| \leq r, |t| \leq \rho_0\}$$

can be covered by balls $B_H(p_1, r), \dots, B_H(p_K, r)$ with $K \leq C(n, b, \rho_0)/r^2$.

Proof. Since $C_r(z_0)$ has height $2\rho_0$ it can be covered by at most $2\rho_0/r$ cubes of side length $2r$ which are parallel to the coordinate axes. (Note that $\frac{\rho_0}{r} \geq 1$ by assumption.) Each of these cubes is contained in a closed Euclidean ball with the same center and radius $\sqrt{2n+1}\,r$. By Lemma 6.4 each of these balls B_i can be covered by at most $C(n, b, \rho_0)/r$ Heisenberg balls of radius r, for some constant $C(n, b, \rho_0)$. The result follows. □

We will also need a covering result of the following type.

Lemma 6.6. *There exists $C > 0$ such that for all $0 < r^2 < \rho < 1$ the set*

$$C_{r,\rho} := \{(z, t) \in \mathbb{H}^n : |z| \leq r, |t| \leq \rho\}$$

can be covered by at most $C\rho/r^2$ balls $B_H(p_1, r), \dots, B_H(p_K, r)$.

Proof. The set $C_{r,\rho}$ can be covered by K parallelepipeds of the form

$$B_\infty((0, t_i), r) := \{(z, t) \in \mathbb{H}^n : |z| \leq r, |t - t_i| \leq r^2\}$$

with $K \leq \frac{2\rho}{r^2}$. (Note that $\frac{\rho}{r^2} > 1$ by assumption.) Since $B_\infty((0, t), r) \subset B_H((0, t), \sqrt[4]{2}r)$, an application of the doubling property of $\mathbb{H}^n$ completes the proof. □

Proof of Proposition 6.1. We begin by observing for $V \in G_h(n, m)$, $u \in V$ and $\delta > 0$ that

$$P_{\mathbb{V}\sharp}\mu(B(u, \delta)) = \mu(P_{\mathbb{V}}^{-1}(B(u, \delta))) = \mu(\{p : |P_{\mathbb{V}}(p) - u| \leq \delta\}).$$

The ball $B(u, \delta) \subset \mathbb{V}$ can equivalently be seen as a ball with respect to Euclidean metric or with respect to the Heisenberg metric as the two distances coincide on $\mathbb{V}$. Using Fatou's lemma and the transformation formula $\int g \, \mathrm{d} f_\sharp \mu = \int (g \circ f) \, \mathrm{d}\mu$ for Borel maps and measures (see e.g. [16, Theorem 1.19]), we compute

$$\iint \liminf_{\delta \to 0} \delta^{-m} P_{\mathbb{V}\sharp}\mu(B(u, \delta)) \, \mathrm{d}P_{\mathbb{V}\sharp}\mu u \, \mathrm{d}\mu_{n,m} V$$
$$\leq \liminf_{\delta \to 0} \delta^{-m} \iint \mu(\{p : |P_{\mathbb{V}}(p) - P_{\mathbb{V}}(q)| \leq \delta\}) \, \mathrm{d}\mu q \, \mathrm{d}\mu_{n,m} V.$$

By the identity

$$\int \mu(\{x : (x, y) \in A\}) \, \mathrm{d}\nu y = \int \nu(\{y : (x, y) \in A\}) \, \mathrm{d}\mu x$$

for a Borel set A and locally finite Borel measures μ and ν on separable metric spaces X and Y (see, e.g., [16, Theorem 1.14]), it follows that

$$\liminf_{\delta \to 0} \delta^{-m} \iint \mu\{p : |P_{\mathbb{V}}(p) - P_{\mathbb{V}}(q)| \leq \delta\} \, \mathrm{d}\mu q \, \mathrm{d}\mu_{n,m} V$$

$$= \liminf_{\delta\to 0} \delta^{-m} \iint \mu_{n,m}(\{V : |P_{\mathbb{V}}(\pi(p)) - P_{\mathbb{V}}(\pi(q))| \le \delta\})\, d\mu q\, d\mu p$$
$$= \liminf_{\delta\to 0} \delta^{-m} \iint \mu_{n,m}(\{V : |\pi_V(\pi(p) - \pi(q))| \le \delta\})\, d\mu q\, d\mu p$$
$$\lesssim \iint |\pi(p) - \pi(q)|^{-m}\, d\mu q\, d\mu p,$$

where we applied Lemma 2.4 in the last line.

Note that instead of an energy integral of the measure μ with respect to the Heisenberg metric d_H, we have obtained the modified energy integral

$$\iint |\pi(p) - \pi(q)|^{-m}\, d\mu q\, d\mu p$$

as an upper bound. We have to make sure that this integral is finite. To this end, let $R > 0$ be large enough so that $\operatorname{spt}\mu \subset B_H(0, R)$. Fix $z \in \mathbb{R}^{2n}$ and $0 < \lambda < 1$. The set $\{q \in \mathbb{H}^n : |\pi(q)-z| \le \lambda\}$ is a cylinder over the Euclidean ball $B_E(z, \lambda)$ with center z and radius λ in $\mathbb{R}^{2n}$. There exists $1 < h(R) < \infty$ such that

$$\{q \in \mathbb{H}^n : |\pi(q) - z| \le \lambda\} \cap \operatorname{spt}\mu \subset B_E(z, \lambda) \times [-h(R), h(R)]. \tag{6.4}$$

By Lemma 6.5 it follows that the cylinder, and thus also the set on the left hand side of (6.4), can be covered by at most $C(n, R)/\lambda^2$ Heisenberg balls of radius λ, where $C(n, R)$ denotes a constant independent of z and λ. Then $\mu(\{q \in \mathbb{H}^n : |\pi(q) - z| \le \lambda\}) \le C(n, R)\lambda^{s-2}$ and

$$\begin{aligned}\int |\pi(p) - z|^{-m}\, d\mu q &= \int_0^\infty \mu(\{q : |\pi(q) - z| \le \eta^{-1/m}\})\, d\eta \\ &= \int_0^1 \mu(\{q : |\pi(q) - z| \le \eta^{-1/m}\})\, d\eta \\ &\quad + \int_1^\infty \mu(\{q : |\pi(q) - z| \le \eta^{-1/m}\})\, d\eta \\ &\le \mu(\mathbb{H}^n) + C(n, R) \int_1^\infty \eta^{(2-s)/m}\, d\eta\end{aligned}$$

which is finite since $s - 2 > m$. We point out that the given upper bound is independent of z. This result can be applied to conclude that

$$\iint \liminf_{\delta\to 0} \delta^{-m} P_{\mathbb{V}\sharp}\mu(B(u, \delta))\, dP_{\mathbb{V}\sharp}\mu u\, d\mu_{n,m} V$$

is finite. Consequently, for $\mu_{n,m}$ a.e. $V \in G_h(n, m)$, we find

$$\liminf_{\delta\to 0} \delta^{-m} P_{\mathbb{V}\sharp}\mu(B(u, \delta)) < \infty \quad \text{for } P_{\mathbb{V}\sharp}\mu \text{ a.e } u \in V.$$

Then (6.3) follows from [16, Theorem 2.12 (3)]. □

By Proposition 6.1, the pushforward measure $P_{\mathbb{V}\sharp}\mu$ is absolutely continuous with respect to the Hausdorff m-measure $\mathcal{H}^m \lfloor \mathbb{V}$ for $\mu_{n,m}$ a.e. $V \in G_h(n, m)$. For any such V and $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$, we can define slice measures $\mu_{V,u}$ as in [16, 10.1] with the properties that

$$\operatorname{spt}\mu_{V,u} \subset \operatorname{spt}\mu \cap \mathbb{V}_u^{\perp} \tag{6.5}$$

and

$$\int_{\mathbb{V}} \mu_{V,u}(\mathbb{H}^n)\,\mathrm{d}\mathcal{H}^m u = \mu(\mathbb{H}^n) \quad \text{if } P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor \mathbb{V}. \tag{6.6}$$

More precisely, we start with a continuous nonnegative compactly supported function φ and define a Radon measure ν_φ by setting

$$\nu_\varphi(A) := \int_A \varphi\,\mathrm{d}\mu$$

for all Borel sets $A \subset \mathbb{H}^n$. As explained in [16, p. 140], it follows that $P_{\mathbb{V}\sharp}\nu_\varphi$ is a Radon measure on $\mathbb{V}$ and that the Radon–Nikodym derivative $D(P_{\mathbb{V}\sharp}\nu_\varphi, \mathcal{H}^m, u)$ exists for $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$. In other words, for $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$, the following limit exists:

$$\mu_{V,u}(\varphi) := \lim_{\delta\downarrow 0}(2\delta)^{-m} P_{\mathbb{V}\sharp}\nu_\varphi(B(u,\delta)) = \lim_{\delta\downarrow 0}(2\delta)^{-m}\int_{P_{\mathbb{V}}{}^{-1}(B(u,\delta))}\varphi\,\mathrm{d}\mu.$$

In the above construction we first fixed φ and then defined $\mu_{V,u}(\varphi)$ by the derivative for $\mathcal{H}^m$ a.e. u. The exceptional set of points u for which the limit does not exist will a priori depend on the choice of φ. However by the separability of $\mathcal{C}_0^+(\mathbb{H}^n)$, one can eliminate the dependence on φ. Thus we can define for $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$ a nonnegative function on $\mathcal{C}_0^+(\mathbb{H}^n)$ by

$$\varphi \mapsto \lim_{\delta\downarrow 0}(2\delta)^{-m}\int_{P_{\mathbb{V}}{}^{-1}(B(u,\delta))}\varphi\,\mathrm{d}\mu.$$

This functional extends to a positive linear functional on $\mathcal{C}_0(\mathbb{H}^n)$ and it follows by the Riesz representation theorem that for $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$ there exists a Radon measure $\mu_{V,u}$ so that

$$\int \varphi\,\mathrm{d}\mu_{V,u} = \lim_{\delta\downarrow 0}(2\delta)^{-m}\int_{P_{\mathbb{V}}{}^{-1}(B(u,\delta))}\varphi\,\mathrm{d}\mu$$

for all $\varphi \in \mathcal{C}_0^+(\mathbb{H}^n)$. This gives immediately

$$\mathrm{spt}\mu_{V,u} \subset \mathrm{spt}\mu \cap P_{\mathbb{V}}^{-1}(u) = \mathrm{spt}\mu \cap \mathbb{V}_u^{\perp}.$$

We call $\mu_{V,u}$ the *slicing measure* associated to the subspace V at the point u.

Because any lower semicontinuous function $g : \mathbb{H}^n \to [0,\infty]$ is the limit of a nondecreasing sequence $\varphi_i \in \mathcal{C}_0^+(\mathbb{H}^n)$ it follows from the above identity that

$$\int g\,\mathrm{d}\mu_{V,u} \le \liminf_{\delta\to 0}(2\delta)^{-m}\int_{P_{\mathbb{V}}{}^{-1}(B(u,\delta))} g\,\mathrm{d}\mu \tag{6.7}$$

for $\mathcal{H}^m$ almost all $u \in \mathbb{V}$.

Now let B be a Borel set in $\mathbb{V}$ and let $\varphi \in \mathcal{C}_0^+(\mathbb{H}^n)$. Theorem 2.12(2) in [16] implies that

$$\int_B D(P_{\mathbb{V}\sharp}\nu_\varphi, \mathcal{H}^m, u)\,\mathrm{d}\mathcal{H}^m u \le P_{\mathbb{V}\sharp}\nu_\varphi(B) = \int_{P_{\mathbb{V}}^{-1}(B)}\varphi\,\mathrm{d}\mu \tag{6.8}$$

with equality if $P_{\mathbb{V}\sharp}\nu_\varphi \ll \mathcal{H}^m$. The left-hand side of (6.8) is equal to

$$\int_B \lim_{\delta\downarrow 0}(2\delta)^{-m}\int_{P_{\mathbb{V}}{}^{-1}(B(u,\delta))}\varphi\,\mathrm{d}\mu\,\mathrm{d}\mathcal{H}^m u = \int_B\int\varphi\,\mathrm{d}\mu_{V,u}\,\mathrm{d}\mathcal{H}^m u,$$

so we conclude for any $\varphi \in \mathcal{C}_0^+(\mathbb{H}^n)$ and any Borel set $B \subset \mathbb{V}$ that

$$\int_B \int \varphi \, \mathrm{d}\mu_{V,u} \, \mathrm{d}\mathcal{H}^m u \le \int_{P_{\mathbb{V}}^{-1}(B)} \varphi \, \mathrm{d}\mu \tag{6.9}$$

with equality if $P_{\mathbb{V}\sharp}\nu_\varphi \ll \mathcal{H}^m$. Since

$$P_{\mathbb{V}\sharp}\nu_\varphi(A) = \int_{P_{\mathbb{V}}^{-1}(A)} \varphi \, \mathrm{d}\mu \le \|\varphi\|_\infty P_{\mathbb{V}\sharp}\mu(A),$$

the absolute continuity statement $P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor \mathbb{V}$ implies $P_{\mathbb{V}\sharp}\nu_\varphi \ll \mathcal{H}^m \lfloor \mathbb{V}$ for all $\varphi \in \mathcal{C}_0^+(\mathbb{H}^n)$. Hence, equality holds in (6.9) for any Borel set B in $\mathbb{V}$ and $\varphi \in \mathcal{C}_0^+(\mathbb{H}^n)$, provided that $P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor \mathbb{V}$. We remind the reader that $\mu_{n,m}$ a.e. V is of this type.

Using again the fact that every lower semicontinuous function on $\mathbb{H}^n$ is a nondecreasing limit of functions in $\mathcal{C}_0^+(\mathbb{H}^n)$ we conclude that (6.9) holds for functions which are merely lower semicontinuous: for each lower semicontinuous $g : \mathbb{H}^n \to [0, \infty]$ we have

$$\int_B \int g \, \mathrm{d}\mu_{V,u} \, \mathrm{d}\mathcal{H}^m(u) \le \int_{P_{\mathbb{V}}^{-1}(B)} g \, \mathrm{d}\mu \tag{6.10}$$

for all Borel sets $B \subset \mathbb{V}$, with equality if $P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor V$. Inserting $B = \mathbb{V}$ and $g \equiv 1$ yields

$$\mu(\mathbb{H}^n) = \int_{\mathbb{V}} \mu_{V,u}(\mathbb{H}^n) \, \mathrm{d}\mathcal{H}^m u \quad \text{if } P_{\mathbb{V}\sharp}\mu \ll \mathcal{H}^m \lfloor \mathbb{V}. \tag{6.11}$$

Next, we provide an analogue of Theorem 10.7 in [16].

Proposition 6.7. *Let s and μ be as in Proposition 6.1 and $\mu_{V,u}$ be a slicing measure defined as above. Then*

$$\iint_{\mathbb{V}} I_{\sigma-m}(\mu_{V,u}) \, \mathrm{d}\mathcal{H}^m u \, \mathrm{d}\mu_{n,m} V < \infty \tag{6.12}$$

for each σ satisfying $m + 2 < \sigma < s$.

Proof. To prove (6.12), we start by inserting the definition of the energy integral,

$$\begin{aligned} I &:= \iint_{\mathbb{V}} I_{\sigma-m}(\mu_{V,u}) \, \mathrm{d}\mathcal{H}^m u \, \mathrm{d}\mu_{n,m} V \\ &= \iint_{\mathbb{V}} \iint d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu_{V,u} p \, \mathrm{d}\mu_{V,u} q \, \mathrm{d}\mathcal{H}^m u \, \mathrm{d}\mu_{n,m} V. \end{aligned}$$

Inequality (6.7), Fatou's lemma and Fubini's theorem yield

$$\begin{aligned} I &\le \liminf_{\delta\to 0} (2\delta)^{-m} \iint_{\mathbb{V}} \iint_{\{p:\, |P_{\mathbb{V}}(p)-u| \le \delta\}} d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu p \, \mathrm{d}\mu_{V,u} q \, \mathrm{d}\mathcal{H}^m u \, \mathrm{d}\mu_{n,m} V \\ &= \liminf_{\delta\to 0} (2\delta)^{-m} \iiint_{\{u\in\mathbb{V}:\, |P_{\mathbb{V}}(p)-u| \le \delta\}} \int d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu_{V,u} q \, \mathrm{d}\mathcal{H}^m u \, \mathrm{d}\mu p \, \mathrm{d}\mu_{n,m} V. \end{aligned}$$

Using inequality (6.10) for $\mu_{n,m}$ almost every $V \in G_h(n,m)$ with $B = \{u : |P_{\mathbb{V}}(p) - u| \le \delta\}$, this can be further estimated:

$$I \le \liminf_{\delta\to 0} (2\delta)^{-m} \iiint_{\{q:\, |P_{\mathbb{V}}(p)-P_{\mathbb{V}}(q)| \le \delta\}} d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu q \, \mathrm{d}\mu p \, \mathrm{d}\mu_{n,m} V.$$

Then by a second application of Fubini's theorem,

$$I \le \liminf_{\delta\to 0}(2\delta)^{-m} \iint \mu_{n,m}(\{V : |P_{\mathbb{V}}(p) - P_{\mathbb{V}}(q)| \le \delta\}) d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu q \, \mathrm{d}\mu p$$
$$\lesssim \iint |\pi(p) - \pi(q)|^{-m} d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu q \, \mathrm{d}\mu p.$$

Here we have again exploited the fact that $P_{\mathbb{V}}$ can be seen as a Euclidean orthogonal projection in the plane and we concluded by applying Lemma 2.4.

In contrast with the Euclidean case, we have obtained here yet another modified energy integral, this time involving the kernel function

$$K(p) = |\pi(p)|^{-m} \, \|p\|_H^{m-\sigma},$$

where $\|\cdot\|_H$ denotes the Korányi norm defined in (1.2). Observe that this kernel presents a stronger singularity than does the kernel for the usual σ-energy, in view of the inequality $|\pi(p)| \le \|p\|_H$. Nevertheless, we will still show that the integral in question is finite. In fact, we claim that there exists a finite constant $C = C(\mu, s, \sigma)$ such that

$$\int |\pi(p) - \pi(q)|^{-m} d_H(p,q)^{m-\sigma} \, \mathrm{d}\mu q \le C \tag{6.13}$$

for all $p \in \mathrm{spt}(\mu)$. Observe that the integrand in (6.13) is invariant under the left translation $L_{p^{-1}} : \mathbb{H}^n \to \mathbb{H}^n$ given by $L_{p^{-1}}(q) = p^{-1} * q$. As a result, it suffices to assume that $p = 0$. The pushforward measure $(L_{p^{-1}})_\sharp \mu$ fulfills the same conditions as μ and has the same total measure. By applying a suitable dilation we can further assume that ν is supported in the closed unit ball $B = B_H(0,1)$.

The preceding discussion indicates that it suffices to prove that

$$\int |\pi(q)|^{-m} \|q\|_H^{m-\sigma} \, \mathrm{d}\nu q \le C$$

for some $C = C(s, \sigma, \nu(\mathbb{H}^n))$ and any Radon measure ν on $\mathbb{H}^n$ supported in B and satisfying the growth condition $\nu(B_H(p,r)) \le r^s$ for all $p \in \mathbb{H}^n$ and $r > 0$. To this end, we compute

$$\int |\pi(q)|^{-m} \|q\|_H^{m-\sigma} \, \mathrm{d}\nu q = \int |\zeta|^{-m} (|\zeta|^4 + \tau^2)^{\frac{m-\sigma}{4}} \, \mathrm{d}\nu(\zeta,\tau)$$
$$\approx \int_{\{|\zeta|^2 \ge |\tau|\}} \|q\|_H^{-\sigma} \, \mathrm{d}\nu(\zeta,\tau) + \int_{\{|\zeta|^2 < |\tau|\}} |\zeta|^{-m} |\tau|^{\frac{m-\sigma}{2}} \, \mathrm{d}\nu(\zeta,\tau).$$

We consider separately the two integrals

$$J_1 := \int_{\{|\zeta|^2 \ge |\tau|\}} \|q\|_H^{-\sigma} \, \mathrm{d}\nu(\zeta,\tau) \quad \text{and} \quad J_2 := \int_{\{|\zeta|^2 < |\tau|\}} |\zeta|^{-m} |\tau|^{\frac{m-\sigma}{2}} \, \mathrm{d}\nu(\zeta,\tau).$$

First,

$$J_1 \le \int_{\mathbb{H}^n} \|q\|_H^{-\sigma} \, \mathrm{d}\nu q = \int_0^\infty \nu(\{q : \|q\|_H^{-\sigma} \ge \lambda\}) \, \mathrm{d}\lambda = \int_0^\infty \nu(B_H(0, \lambda^{-\frac{1}{\sigma}})) \, \mathrm{d}\lambda$$
$$= \sigma \int_0^1 \nu(B_H(0,r)) r^{-\sigma-1} \, \mathrm{d}r + \sigma \int_1^\infty \nu(B_H(0,r)) r^{-\sigma-1} \, \mathrm{d}r$$
$$\le \sigma \int_0^1 r^{s-\sigma-1} \, \mathrm{d}r + \nu(\mathbb{H}^n)\sigma \int_1^\infty r^{-\sigma-1} \, \mathrm{d}r =: C_1(s, \sigma, \nu(\mathbb{H}^n)) < \infty.$$

Concerning the second integral, we decompose

$$\{(\zeta,\tau)\in B:\ 0<|\zeta|^2<|\tau|\}=\bigcup_{i=0}^{\infty}\{(\zeta,\tau)\in B:\ 2^{-i-1}|\tau|\le|\zeta|^2<2^{-i}|\tau|\}.$$

For $2^{-i-1}|\tau|\le|\zeta|^2<2^{-i}|\tau|$ we have $|\zeta|^{-1}\approx 2^{\frac{i}{2}}|\tau|^{-1/2}$. Hence

$$\begin{aligned}
J_2 &\approx \sum_{i=0}^{\infty}\int_{\{2^{-i-1}|\tau|\le|\zeta|^2<2^{-i}|\tau|\}} \left(2^{\frac{i}{2}}|\tau|^{-\frac{1}{2}}\right)^m|\tau|^{\frac{m-\sigma}{2}}\,\mathrm{d}\nu(\zeta,\tau)\\
&= \sum_{i=0}^{\infty}\int_{\{2^{-i-1}|\tau|\le|\zeta|^2<2^{-i}|\tau|\}} 2^{\frac{im}{2}}|\tau|^{-\frac{\sigma}{2}}\,\mathrm{d}\nu(\zeta,\tau)\\
&\approx \sum_{i,j=0}^{\infty}\int_{\left\{\substack{2^{-i-j-2}\le|\zeta|^2<2^{-i-j}\\ 2^{-j-1}\le|\tau|<2^{-j}}\right\}} 2^{\frac{im}{2}}\left(2^{-j}\right)^{-\frac{\sigma}{2}}\,\mathrm{d}\nu(\zeta,\tau)\\
&= \sum_{i,j=0}^{\infty}2^{\frac{im}{2}+\frac{\sigma j}{2}}\nu(A_{i,j}),
\end{aligned}$$

with

$$A_{i,j}=\{(\zeta,\tau):\ 2^{-(i+j+2)/2}\le|\zeta|<2^{-(i+j)/2},\ 2^{-j-1}\le|\tau|<2^{-j}\}.$$

We cover $A_{i,j}$ by balls of a fixed radius comparable to $2^{-(i+j)/2}$ and use the growth condition on ν to estimate $\nu(A_{i,j})$ from above. We slightly enlarge the set $A_{i,j}$ by taking the cylinder over a ball instead of the cylinder over an annulus. The number of balls needed to cover this set can be estimated by Lemma 6.6. Applying this result with $\rho=2^{-j}$ and $r=2^{-(i+j)/2}$ it follows that $A_{i,j}$ can be covered by at most $C\frac{\rho}{r^2}=2^iC$ balls of radius $2^{-(i+j)/2}$, where C is independent of r and ρ. In this case, the growth condition on ν implies that

$$\nu(A_{i,j})\lesssim 2^i\left(2^{-(i+j)/2}\right)^s=2^{i-s(i+j)/2}.$$

Therefore,

$$J_2\lesssim\sum_{i,j=0}^{\infty}2^{\frac{im}{2}+\frac{\sigma j}{2}+i-\frac{s(i+j)}{2}}=\sum_{i,j=0}^{\infty}2^{i\left(\frac{-s+(m+2)}{2}\right)}2^{j\left(\frac{\sigma-s}{2}\right)}\le C_2<\infty,$$

since $s>m+2$ and $s>\sigma$. Note that C_1 and C_2 depend only on s, σ, m and $\operatorname{diam}(\operatorname{spt}\mu)$. This proves (6.13) and hence (6.12). □

We are now prepared to prove Theorem 1.5.

Proof of Theorem 1.5. First, we prove the universal upper bound

$$\dim_H(A\cap\mathbb{V}_u^{\perp})\le\dim_H A-m \tag{6.14}$$

for all $V\in G_h(n,m)$ and $\mathcal{H}^m$ almost all $u\in\mathbb{V}$. Since $P_{\mathbb{V}}:(\mathbb{H}^n,d_H)\to\mathbb{V}$ is Lipschitz, it follows from [8, 2.10.25] that

$$\int\mathcal{H}_H^{s-m}(A\cap\mathbb{V}_u^{\perp})\,\mathrm{d}\mathcal{H}^m u=\int\mathcal{H}_H^{s-m}\left(A\cap P_{\mathbb{V}}^{-1}(u)\right)\mathrm{d}\mathcal{H}^m u\lesssim\mathcal{H}_H^s(A).$$

If $s > \dim_H A$ then $\mathcal{H}^s_H(A) = 0$ and the integrand $\mathcal{H}^{s-m}_H(A \cap \mathbb{V}_u^\perp)$ is zero for $\mathcal{H}^m$ almost every $u \in \mathbb{V}$. We conclude that $\dim_H(A \cap \mathbb{V}_u^\perp) \le s - m$ from which one obtains (6.14).

Next we prove (6.1). Denote $s = \dim_H A > m + 2$. Then there exists a measure μ which fulfills the assumptions of Proposition 6.1. For $V \in G_h(n,m)$ denote

$$E_V = \{u \in \mathbb{V} : \mu_{V,u} \text{ is defined and } \mu_{V,u}(\mathbb{H}^n) > 0\}.$$

For $\mu_{n,m}$ a.e. $V \in G_h(n,m)$ we have by (6.11) and Proposition 6.1 that $\mathcal{H}^m(E_V) > 0$, and further by Proposition 6.7 that for any $m + 2 < \sigma < s$, $I_{\sigma-m}(\mu_{V,u})$ is finite for $\mathcal{H}^m$ a.e. $u \in E_V$. Then since $\mu_{V,u}$ is supported on $\operatorname{spt}\mu \cap \mathbb{V}_u^\perp \subset A \cap \mathbb{V}_u^\perp$ (see (6.5)), this implies that $\dim_H(A \cap \mathbb{V}_u^\perp) \ge \sigma - m$ for $\mathcal{H}^m$ a.e. $u \in E_V$. As $\sigma < s$ was chosen arbitrarily, this gives $\dim_H(A \cap \mathbb{V}_u^\perp) \ge \dim_H A - m$ for $\mathcal{H}^m$ a.e. $u \in E_V$ and completes the proof. □

We conclude this section with an alternate version of the slicing theorem which is closely related to Theorem 1.5.

Theorem 6.8. *Let $A \subset \mathbb{H}^n$ be a Borel set with $0 < \mathcal{H}^s_H(A) < \infty$ for some $s > m + 2$. Then for $\mathcal{H}^s_H$ a.e. $p \in A$ we have $\dim_H(A \cap \mathbb{V}_p^\perp) = s - m$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$.*

Note here that we have used the notation $\mathbb{V}_p^\perp$ for points p which are not necessarily in $\mathbb{V}$.

Proof. First, we prove that $\dim_H(A \cap \mathbb{V}_p^\perp) \ge s - m$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$ and $\mathcal{H}^s_H$ a.e. $p \in A$. Suppose that this conclusion fails to be true. Then there exists σ with $m+2 < \sigma < s$ and a compact set $F \subset A$ so that $\mathcal{H}^s_H(F) > 0$ and $\mu_{n,m}(\{V : \dim_H(A \cap \mathbb{V}_p^\perp) < \sigma - m\})$ is positive for all $p \in F$. By Frostman's lemma, there exists a measure $\mu \in \mathcal{M}(F)$ so that $\mu(B_H(p,r)) \le r^s$ for all $p \in \mathbb{H}^n$ and all $r > 0$. By Fubini's theorem (the measurability assumption is easily verified in this setting due to the compactness of F),

$$\int \mu(\{p : \dim_H(F \cap \mathbb{V}_p^\perp) < \sigma - m\})\, d\mu_{n,m} V$$
$$= \int \mu_{n,m}(\{V : \dim_H(F \cap \mathbb{V}_p^\perp) < \sigma - m\})\, d\mu p > 0.$$

Hence there exists a compact set $G \subset G_h(n,m)$ so that $\mu_{n,m}(G) > 0$ and

$$\mu(\{p : \dim_H(F \cap \mathbb{V}_p^\perp) < \sigma - m\}) > 0 \quad \text{for all } V \in G.$$

By Proposition 6.1, $(P_{\mathbb{V}})_\sharp \mu \ll \mathcal{H}^m$ for $\mu_{n,m}$ a.e. V, and so

$$\mathcal{H}^m(\{u : \dim_H(F \cap \mathbb{V}_u^\perp) < \sigma - m\}) > 0 \quad \text{for } \mu_{n,m} \text{ a.e. } V \in G. \tag{6.15}$$

Here we used the elementary fact that $\mathbb{V}_p^\perp = \mathbb{V}_u^\perp$ for every p so that $P_{\mathbb{V}}(p) = u$.

Now the slicing measure $\mu_{V,u}$ is defined for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$ and $\mathcal{H}^m$ a.e. $u \in \mathbb{V}$ as a Radon measure on $F \cap \mathbb{V}_u^\perp$. By Proposition 6.7,

$$\int \int_V I_{\sigma-m}(\mu_{V,u})\, d\mathcal{H}^m u\, d\mu_{n,m} V < \infty. \tag{6.16}$$

But $I_{\sigma-m}(\mu_{V,u}) = \infty$ when $\mu_{V,u}(\mathbb{H}^n) > 0$ and $\dim_H(F \cap \mathbb{V}_u^\perp) < \sigma - m$. By (6.15) and (6.11)

$$\int_G \int_V I_{\sigma-m}(\mu_{V,u})\, d\mathcal{H}^m u\, d\mu_{n,m} V = \infty,$$

which contradicts (6.16) and consequently finishes this part of the proof.

Next, we prove that $\dim_H(A \cap \mathbb{V}_p^{\perp}) \leq s-m$ for $\mu_{n,m}$ a.e. $V \in G_h(n,m)$ and $\mathcal{H}_H^s$ a.e. $p \in \mathbb{H}^n$. The argument is a modification of the proof of [15, Lemma 6.5]. It suffices to show

$$\int^* \mathcal{H}_H^{s-m}(A \setminus C(p,r) \cap \mathbb{V}_p^{\perp})\, d\mu_{n,m} V < \infty$$

for all $r > 0$, where $C(p,r) = \{q : |\pi(p) - \pi(q)| \leq r\}$. Left translating by p^{-1} and using the fact that $\mathbb{V}^{\perp}$ is a normal subgroup, we may assume that $p = 0$.

For each $k \in \mathbb{N}$ choose balls $B_{k,j}$ satisfying $A \setminus C(0,r) \subset \bigcup_j B_{k,j} \subset \mathbb{H}^n \setminus C(0,r/2)$, $\sum_j (\operatorname{diam} B_{k,j})^s < \mathcal{H}_H^s(A) + 1$, and $\operatorname{diam} B_{k,j} < \frac{1}{k}$. Let $p_{k,j}$ be the center of $B_{k,j}$. Then $|\pi(p_{k,j})| > \frac{r}{2}$. By Lemma 2.4,

$$\begin{aligned}\mu_{n,m}(\{V \in G_h(n,m) : \mathbb{V}^{\perp} \cap B_{k,j} \neq \emptyset\}) &\leq \mu_{n,m}(\{V : |\pi_V(\pi(p_{k,j}))| \leq \operatorname{diam} B_{k,j}\}) \\ &\leq C\left(\frac{r}{2}\right)^{-m} (\operatorname{diam} B_{k,j})^m.\end{aligned}$$

Hence $\int \operatorname{diam}(B_{k,j} \cap \mathbb{V}_p^{\perp})^{s-m}\, d\mu_{n,m} V \lesssim r^{-m} (\operatorname{diam} B_{k,j})^s$. Applying Fatou's lemma yields

$$\begin{aligned}\int^* \mathcal{H}_H^{s-m}(A \setminus C(0,r) \cap \mathbb{V}^{\perp})\, d\mu_{n,m} V &\leq \int \liminf_{k\to\infty} \sum_j (\operatorname{diam} B_{k,j} \cap \mathbb{V}^{\perp})^{s-m}\, d\mu_{n,m} V \\ &\leq \liminf_{k\to\infty} \sum_j \int (\operatorname{diam} B_{k,j} \cap \mathbb{V}^{\perp})^{s-m}\, d\mu_{n,m} V \\ &\lesssim r^{-m} \sum_j (\operatorname{diam} B_{k,j})^s \leq r^{-m}(\mathcal{H}_H^s(A) + 1)\end{aligned}$$

which is finite by assumption. The proof is complete. □

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