SIZE OF TANGENCIES TO NON-INVOLUTIVE DISTRIBUTIONS

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Abstract. By the classical Frobenius Theorem, a distribution is completely integrable if and only if it is involutive. In this paper, we investigate the size of tangencies of submanifolds with respect to a given *non-involutive* distribution. We provide estimates for the size of tangencies in terms of its Hausdorff dimension. This generalises earlier works by Derridj and the first author. Our results apply in the setting of contact and symplectic structures as well as of Carnot groups. We illustrate the sharpness of our estimates by a wide range of examples and round the paper off with additional comments and open questions.

1. Introduction

The classical Frobenius Theorem [2, 20, 24] states that a C^1 smooth distribution is completely integrable if and only if it is involutive. In this paper, we are going to investigate non-involutive distributions $\mathcal{D}$ in terms of their degree of integrability. More precisely, given a non-integrable distribution $\mathcal{D}$ of rank n on an $(n+m)$-dimensional manifold M and an n-dimensional submanifold $S \subseteq M$, we shall investigate the size of the tangency set of S with respect to $\mathcal{D}$. This problem becomes of interest for Carnot-Carathéodory or sub-Riemannian geometries [14] defined by bracket generating distributions $\mathcal{D}$, in particular for contact manifolds [1], Carnot groups [7, 13, 17] or Hörmander vector fields [9, 19]. In all the works mentioned above, the tangency set of a given submanifold S with respect to $\mathcal{D}$ was a pathological set where the methods of sub-Riemannian potential and geometric measure theory failed to work. It is therefore important to show that the tangency set is a small or even a negligible set in terms of Hausdorff measure and dimension.

The main goal of the present paper is to prove an upper bound for the size of the tangency set with respect to a general non-integrable distribution in terms of their Hausdorff dimension. Such statements have been shown earlier by the first author [3] and subsequently applied in [5, 6, 8, 23] to develop the theory of minimal and area stationary surfaces of the Heisenberg group. The result of [3] as well as a classical theorem of Derridj [9] will follow as elegant (and rather straightforward) consequences of our main Theorem 1.3 below.

We shall begin with introducing the necessary notations to formulate our main result. For notational simplicity, we shall assume that the ambient manifold M is the Euclidean space $\mathbb{R}^{n+m}$. Recall [20] that a C^r, $r \in \mathbb{N}$, smooth distribution $\mathcal{D}$ of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ is a C^r smooth assignment to each point $z \in U$ of a linear n-dimensional subspace $\mathcal{D}(z) \preceq T_z(\mathbb{R}^{n+m})$. $\mathcal{D}$ is usually described either by n pointwise linearly independent C^r smooth vector fields

$$\{X_1, \dots, X_n\} \tag{1.1}$$

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such that $\{X_1(z),\ldots,X_n(z)\}$ forms a basis of $\mathcal{D}(z)$ for all $z\in U$ or as the intersection of the kernels of m linearly independent one-forms $\{\vartheta^1,\ldots,\vartheta^m\}$ with C^r smooth coefficients on U, i.e.

$$\mathcal{D}=\ker(\vartheta^1)\cap\ldots\cap\ker(\vartheta^m).$$

Definition 1.1. Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U\subseteq\mathbb{R}^{n+m}$ and $S\subseteq U$ a C^1 smooth n-dimensional manifold. We call a point $z\in S$ a *tangency point* of S with respect to $\mathcal{D}$ if and only if $T_z(S)=\mathcal{D}(z)$. The set of such points is called the *tangency set*, or, in short, the *tangency*, of S with respect to $\mathcal{D}$ and denoted by

$$\tau(S,\mathcal{D}):=\{z\in S: T_z(S)=\mathcal{D}(z)\}.$$

The purpose of this paper is to give a general upper estimate of the Hausdorff dimension $\dim_{\mathrm{H}}(\tau(S,\mathcal{D}))$ of the tangency set $\tau(S,\mathcal{D})$.

Let us denote by $G(n{+}m,k)$ the Grassmannian [18] of k-planes in $\mathbb{R}^{n+m}$. The following definition is crucial for the formulation of our main result.

Definition 1.2. Let $\mathcal{D}=\ker(\vartheta^1)\cap\ldots\cap\ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U\subseteq\mathbb{R}^{n+m}$. For $1\le k\le n+m$, we define

$$\begin{aligned}A_k:=\{z\in U\quad:\quad&\text{there exists } X\in G(n+m,k)\text{ such that}\\ &\vartheta^i_z|_X=0\text{ and } \mathrm{d}\vartheta^i_z|_X=0\text{ for all }1\le i\le m\}.\end{aligned}$$

Clearly, the sets $(A_k)_k$ form a decreasing sequence: $A_1\supseteq A_2\supseteq\ldots\supseteq A_{n+m}$. Note that $A_1=U$ due to the integrability of vector fields [2]. Furthermore, $A_l=\emptyset$ for all $n+1\le l\le n+m$, since the contrary would imply $\mathrm{rank}(\mathcal{D})>n$.

By the Hausdorff dimensions of the members of the sequence $(A_k)_k$, one can control the measure of integrability of the distribution $\mathcal{D}$ as formulated in our main

Theorem 1.3. *Let $\mathcal{D}=\ker(\vartheta^1)\cap\ldots\cap\ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U\subseteq\mathbb{R}^{n+m}$, A_k defined as above, $S\subseteq U$ an n-dimensional C^2 smooth manifold and $\tau(S,\mathcal{D})$ its tangency set with respect to $\mathcal{D}$. Then*

$$\dim_{\mathrm{H}}(\tau(S,\mathcal{D}))\le\max_{1\le k\le n}\{\min\{\dim_{\mathrm{H}}(A_k\setminus A_{k+1}),k\}\}.\tag{1.2}$$

The following example illustrates the sharpness of our theorem and shows that the maximal possible Hausdorff dimension of a tangency set does not need to be integer.

Example 1.4. Let $U:=(0,1)^2\times(-1,1)\subseteq\mathbb{R}^3$ and define the distribution $\mathcal{D}$ of rank 2 on U as follows: Let $C\subseteq(0,1)$ denote a compact set of dimension $s\in(0,1)$ and consider the set $A:=(0,1)\times C\times\{0\}\subseteq U$. A will serve as the 'critical' A_2, which realises the maximum in (1.2). By an application of Whitney decomposition and partition of unity [16, Proposition 2.3.4], there exists a C^∞ smooth function $\varphi:\mathbb{R}^3\to\mathbb{R}^+_0$ whose zero set is precisely A. Define, for any $z=(x,y,t)\in U$, the real-valued function

$$f(x,y,t):=\int_0^x\varphi(s,y,t)\,\mathrm{d}s$$

as well as the C^∞ smooth pointwise defined one-form $\vartheta_z:=f(z)\cdot\mathrm{d}y-\mathrm{d}t$.

It is immediate to see that the corresponding distribution is pointwise defined by $\mathcal{D}(z)=\ker(\vartheta_z)=\mathrm{span}\{\partial_x,\partial_y+f(z)\cdot\partial_t\}$ and that $\mathrm{d}\vartheta_z=f_x(z)\cdot\mathrm{d}x\wedge\mathrm{d}y-f_t(z)\cdot\mathrm{d}y\wedge\mathrm{d}t$. Together with $\mathrm{d}\vartheta_z(\partial_x,\partial_y+f(z)\cdot\partial_t)=f_x(z)$, we get

$$A_2=\{z\in U:\mathrm{d}\vartheta|_{\ker(\vartheta)}=0\}=\{z\in U: f_x(z)=0\}=A.$$

Since $A_3=\emptyset$, this implies

$$\dim_{\mathrm{H}}(\tau(S,\mathcal{D}))\le\min\{\dim_{\mathrm{H}}(A_2\setminus A_3),2\}=1+\dim_{\mathrm{H}}(C)=1+s\tag{1.3}$$

for any C^2 smooth 2-dimensional manifold $S\subseteq U$.

We also point out that the estimate (1.3) is sharp: Let $S := \{z \in U : t = 0\}$. Then

$$\begin{aligned}\tau(S, \mathcal{D}) &= \{z = (x, y, 0) : \mathcal{D}(z) = \mathrm{span}\{\partial_x, \partial_y\}\} \\ &= \{z = (x, y, 0) : f(x, y, 0) = 0\} = A\end{aligned}$$

by the definition of the function f and therefore $\dim_{\mathrm{H}}(\tau(S, \mathcal{D})) = 1 + s$.

In spite of its abstract appearance, the above result has sharp applications in several cases of important distributions. It turns out that the sets A_k appearing on the right side of (1.2) are large for small values of k. However, a sudden drop of the size of A_k can occur at some critical value. And this critical value k_0 is the one to maximise the right side of (1.2). In particular, applying the above theorem, we obtain, in a rather elegant way, a generalisation of a classical result of Derridj [9] stating that the set of tangencies of hypersurfaces to distributions satisfying the Hörmander condition is negligible. Similarly, the result of the first author [3] on the size of characteristic sets of surfaces in contact manifolds (in particular in the Heisenberg group) follows directly from Theorem 1.3. Let us mention that the C^2 smoothness of S is a necessary assumption. We shall show in Section 8 that for $C^{1,\alpha}$, $0 < \alpha < 1$, regular manifolds all control over the size of the tangency set is lost in general.

The paper is organised as follows: In the second section, we introduce the notion of the tangent space to a general subset of the Euclidean space and show that its algebraic dimension controls the Hausdorff dimension of the set itself from above. Section 3 is devoted to the proof of our main Theorem 1.3 above. In Section 4, we state and prove a generalised version of Derridj's Theorem [9]. Lower estimates for the size of tangencies are presented in Section 5. In Section 6, we apply Theorem 1.3 to the important cases of contact and symplectic structures and in Carnot groups. In particular, we obtain the main result of [3] as a direct consequence. We investigate the sharpness of our main theorem in Section 7. In the final Section 8, we discuss the sharpness of Theorem 1.3 in terms of its regularity assumptions and round the article off with additional comments and open questions.

2. Tangent space to a general set

Before turning our attention to the main statements of this paper, we state and prove preliminary results, which will be useful in the sequel. In this direction, we start with the following

Definition 2.1. Let $A \subseteq \mathbb{R}^n$ and $x \in A$. We define the *set of tangent directions* of A at x as

$$\mathrm{Dir}_x(A) := \left\{ v \in S^{n-1} : \exists (x_n) \text{ in } A, x_n \to x, x_n \neq x, \text{ such that } \tfrac{x_n - x}{|x_n - x|} \to v \right\}.$$

In addition, we define the *tangent space* of A at x as $\mathrm{Tan}_x(A) := \mathrm{span}(\mathrm{Dir}_x(A))$.

Proposition 2.2. $\mathrm{Dir}_x(A)$ *and* $\mathrm{Tan}_x(A)$ *have natural properties: Let* $A, B \in \mathbb{R}^n$.

1) *If* $x \in A \subseteq B$ *then* $\mathrm{Dir}_x(A) \subseteq \mathrm{Dir}_x(B)$ *and* $\mathrm{Tan}_x(A) \preceq \mathrm{Tan}_x(B)$.
2) *The following statements are equivalent:*
   a) *x is an isolated point of A;*
   b) $\mathrm{Dir}_x(A) = \emptyset$*;*
   c) $\mathrm{Tan}_x(A) = \{0\}$.
3) *Let $U \subseteq \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^m$ a C^1 smooth embedding, $A \subseteq U$ and $x \in A$. Then* $(\mathrm{D}f(x))(\mathrm{Tan}_x(A)) = \mathrm{Tan}_{f(x)}(f(A))$.

Proof. While 1) and 2) are obvious, 3) is slightly more intricate; we therefore only give a proof of 3), assuming without loss of generality that x is not an isolated point of A. To prove the inclusion $(\mathrm{D}f(x))(\mathrm{Tan}_x(A)) \subseteq \mathrm{Tan}_{f(x)}(f(A))$ consider $u \in \mathrm{Dir}_x(A)$ and a sequence (x_n) in A with $x_n \to x$, $x_n \neq x$ and $\frac{x_n - x}{|x_n - x|} \to u$. If $(\mathrm{D}f(x))(u) = 0$ there is nothing to prove. For the case $(\mathrm{D}f(x))(u) \neq 0$ note that

$$\frac{f(x_n) - f(x)}{|x_n - x|} = \frac{f(x_n) - f(x) - (\mathrm{D}f(x))(x_n - x)}{|x_n - x|} + (\mathrm{D}f(x))\left(\frac{x_n - x}{|x_n - x|}\right)$$

converges to $(\mathrm{D}f(x))(u)$, which implies that

$$\frac{f(x_n) - f(x)}{|f(x_n) - f(x)|} = \left|\frac{x_n - x}{f(x_n) - f(x)}\right| \cdot \frac{f(x_n) - f(x)}{|x_n - x|} \to \frac{(\mathrm{D}f(x))(u)}{|(\mathrm{D}f(x))(u)|}. \tag{2.1}$$

Therefore $\frac{(\mathrm{D}f(x))(u)}{|(\mathrm{D}f(x))(u)|} \in \mathrm{Dir}_{f(x)}(f(A))$, which yields the first inclusion.

For the opposite inclusion consider $v \in \mathrm{Dir}_{f(x)}(f(A))$ and a sequence (y_n) in $f(A)$, $y_n \to y$, $y_n \neq y$, such that $\frac{y_n - y}{|y_n - y|} \to v$. Since $y_n \in f(A)$ for all $n \in \mathbb{N}$ and f is an embedding, there exists a sequence (x_n) in A, $x_n \to x$, $x_n \neq x$, such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $y = f(x)$. By choosing a subsequence, if necessary, we can assume that $\left(\frac{x_n - x}{|x_n - x|}\right)$ converges to some $u \in S^{n-1}$. Then $u \in \mathrm{Dir}_x(A)$ and (2.1) implies that

$$v = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{|f(x_n) - f(x)|} = \frac{(\mathrm{D}f(x))(u)}{|(\mathrm{D}f(x))(u)|} \in \mathrm{span}((\mathrm{D}f(x))(\mathrm{Dir}_x(A))),$$

which shows the opposite inclusion. $\square$

Remark 2.3. If the condition f to be an embedding in Proposition 2.2 3) is omitted, then still $(\mathrm{D}f(x))(\mathrm{Tan}_x(A)) \subseteq \mathrm{Tan}_{f(x)}(f(A))$.

We next estimate the Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ in terms of the algebraic dimension of its tangent space. Before we can state the main result in this direction, we need additional notation and the following lemma, which is a generalisation of [18, Lemma 15.13].

Following [18] we denote by $G(n,k)$ the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, where $1 \leq k \leq n$. For $V \in G(n,k)$ we denote by $P_V : \mathbb{R}^n \to V$ the orthogonal projection onto V and by $Q_V = P_{V^\perp} : \mathbb{R}^n \to V^\perp$ the orthogonal projection onto the orthogonal complement $V^\perp$. $G(n,k)$ is a compact metric space with the metric $d(V,W) = \|P_V - P_W\|$, where $\|P_V - P_W\|$ stands for the operator norm of the linear map $P_V - P_W : \mathbb{R}^n \to \mathbb{R}^n$. The balls in this metric are denoted by $B(V,r)$ for $V \in G(n,k)$ and $r > 0$. For $x \in \mathbb{R}^n$ and $0 < s < 1$ we define the cone around $V \in G(n,k)$ with vertex x as

$$X(x,V,s) := \{y \in \mathbb{R}^n : \mathrm{dist}(y-x, V) < s\cdot|y-x|\} = \{y \in \mathbb{R}^n : |Q_V(y-x)| < s\cdot|y-x|\}.$$

We recall [18] that a set $A \subseteq \mathbb{R}^n$ is called *k-rectifiable* if there exist subsets $A_i \subseteq \mathbb{R}^k$ and Lipschitz maps $f_i : A_i \to \mathbb{R}^n$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^k\left(A \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$

With $\mathcal{H}^k(A)$, $A \in \mathbb{R}^n$, we denote the k-dimensional Hausdorff measure [11] of A.

Note that the A_i can be replaced by $\mathbb{R}^k$ in the above definition, since there exist Lipschitz extensions of the f_i to the whole space $\mathbb{R}^k$ by Kirszbraun's Theorem [12]. Note also that, by definition, k-rectifiable sets have their Hausdorff dimension less than or equal to k.

Lemma 2.4. *Let $A \subseteq \mathbb{R}^n$ and $0 < s < 1$. If for $\mathcal{H}^k$-almost all $x \in A$ there exist a linear subspace $W = W(x) \in G(n, n-k)$ and $0 < r = r(x) < 1$ such that*

$$A \cap B(x,r) \cap X(x,W,s) = \emptyset, \tag{2.2}$$

then A is k-rectifiable.

Remark 2.5. Note that if (2.2) holds for an $r > 0$, then also for all $0 < r' \leq r$.

Remark 2.6. A similar statement is formulated in [18, Theorem 15.19] with the additional assumption $\mathcal{H}^k(A) < \infty$.

Proof. We shall divide the part of A for which (2.2) holds into a countable collection of subsets A_{ij}, which will be Lipschitz images of suitable subsets of $\mathbb{R}^k$: First note that by the compactness of $G(n, n-k)$ there is a number $N = N(s) \in \mathbb{N}$ and there are subspaces $W_i \in G(n, n-k)$, $1 \leq i \leq N$, such that $\bigcup_{i=1}^N B(W_i, \frac{s}{2}) = G(n, n-k)$. Define

$$A_{ij} := \{x \in A : W(x) \in B(W_i, \tfrac{s}{2}) \text{ and } \tfrac{1}{j+1} \leq r(x) < \tfrac{1}{j}\},$$

where $1 \leq i \leq N$ and $j \in \mathbb{N}$. We can further subdivide the sets A_{ij} into countably many subsets $A_{ijk} \subseteq A_{ij}$ such that $\bigcup_k A_{ijk} = A_{ij}$ and $\operatorname{diam}(A_{ijk}) < \frac{1}{j+1}$. Since the new family is still countable, for notational simplicity we may assume that $\operatorname{diam}(A_{ij}) < \frac{1}{j+1}$.

Let $x, y \in A_{ij}$. Then $|y - x| < \frac{1}{j+1} \leq r(x)$ and by (2.2) $y \notin X(x, W(x), s)$, or, equivalently, $|Q_{W(x)}(y-x)| \geq s \cdot |y - x|$. Since $P_W + Q_W = \mathrm{id}$, we have

$$(P_{W(x)} - P_{W_i} + Q_{W(x)} - Q_{W_i})(y - x) = 0$$

and thus $|(Q_{W(x)} - Q_{W_i})(y-x)| < \frac{s}{2} \cdot |y - x|$. Therefore, by the triangle inequality,

$$|Q_{W_i}(y-x)| \geq |Q_{W(x)}(y-x)| - |(Q_{W(x)} - Q_{W_i})(y-x)| \geq \tfrac{s}{2} \cdot |y - x|.$$

Hence $Q_{W_i}|_{A_{ij}}$ is one-to-one with Lipschitz inverse $f_i = (Q_{W_i}|_{A_{ij}})^{-1}$ with (uniform) Lipschitz constant $L \leq \frac{2}{s}$. Since $Q_{W_i}(A_{ij})$ lies on the k-plane $W_i^{\perp}$ and we have $A_{ij} = f_i(Q_{W_i}(A_{ij}))$, the set A_{ij} – and therefore also A – is k-rectifiable. □

Lemma 2.4 will be of use in the proof of the following

Proposition 2.7. *Let $A \subseteq \mathbb{R}^n$ and $\dim(\operatorname{Tan}_x(A)) \leq k$ for $\mathcal{H}^k$-almost all $x \in A$. Then A is k-rectifiable. In particular $\dim_{\mathrm{H}}(A) \leq k$.*

Proof. Fix $0 < s < 1$. We will show that for $\mathcal{H}^k$-almost all $x \in A$ there exists a k-dimensional subspace $V = V(x) \in G(n, k)$ and $0 < r = r(x) < \infty$ such that

$$(A \cap B(x, r)) \setminus X(x, V, s) = \emptyset.$$

Let $x \in A$. Without loss of generality, we may assume that $\dim(\operatorname{Tan}_x(A)) = k$ and set $V = V(x) = \operatorname{Tan}_x(A)$. Assume by contradiction that for all $r > 0$ there exists

$$x_r \in (A \cap B(x, r)) \setminus X(x, V, s).$$

Choosing $r_n = \frac{1}{n}$ induces a sequence (x_n), $x_n \neq x$, $x_n \to x$, for which

$$x_n \in (A \cap B(x, \tfrac{1}{n})) \setminus X(x, V, s).$$

From $x_n \notin X(x, V, s)$ follows $d(\frac{x_n - x}{|x_n - x|}, V) \geq s$ for all $n \in \mathbb{N}$. By choosing a subsequence, if necessary, we find some $v = \lim_{n\to\infty} \frac{x_n - x}{|x_n - x|} \in \operatorname{Tan}_x(A)$ for which $d(v, V) \geq s$. But this implies $v \notin V$, which contradicts the assumption $v \in \operatorname{Tan}_x(A)$.

Now the claim of the proposition is an immediate consequence of Lemma 2.4 by setting $W(x) := V^{\perp}(x)$ for almost all $x \in A$. □

3. Proof of Theorem 1.3

Following up the discussion in the introduction, we are going to investigate the role, the one-forms defining a distribution and their differentials play to control the size of tangencies. These insights will enable us to prove our main theorem thereafter.

In this direction, first note [20, Proposition 2.11.7] that the vector fields from (1.1) can – after a change of the coordinates, if necessary – always be written in the form

$$X_i(z) = \partial_{x_i} + \sum_{j=1}^{m} c_{ij}(z)\cdot\partial_{y_j} \tag{3.1}$$

for all $z = (x, y) \in U \subseteq \mathbb{R}^n \oplus \mathbb{R}^m$, where $\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_m}\}$ denotes the standard basis of $T_z(\mathbb{R}^{n+m}) = T_x(\mathbb{R}^n) \oplus T_y(\mathbb{R}^m)$. The corresponding one-forms with $\mathcal{D}$ being the intersection of their kernels are

$$\vartheta^j(z) := \sum_{i=1}^{n} c_{ij}(z)\cdot \mathrm{d}x_i - \mathrm{d}y_j, \tag{3.2}$$

$1 \le j \le m$, as can immediately be verified.

We call [20] a distribution $\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_n\}$ of rank n on $U \subseteq \mathbb{R}^{n+m}$ *involutive* if it is closed under the Lie bracket operation, i.e. if for any two vector fields X_i, X_j, $1 \le i, j \le n$, we have $[X_i, X_j](z) \in \mathcal{D}(z)$ for all $z \in U$. In terms of one-forms [2, Theorem 3.8], the distribution $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ is involutive if and only if $\mathrm{d}\vartheta^i \wedge \vartheta^1 \wedge \ldots \wedge \vartheta^m = 0$ for all $1 \le i \le m$. Equivalently, according to [20, 2.11.12], $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ is involutive if and only if there exist one-forms α^{ij} on U such that

$$\mathrm{d}\vartheta^i = \sum_{j=1}^{m} \vartheta^j \wedge \alpha^{ij} \tag{3.3}$$

for all $1 \le i \le m$.

As already pointed out in the introduction, a C^1 smooth distribution is completely integrable if and only if it is involutive by virtue of Frobenius' Theorem [2, 20, 24]. Although the latter fails in our context of *non-involutive* distributions, the tangency set $\tau(S, \mathcal{D})$ from Definition 1.1 can be interpreted as a link to the involutive case in the sense that integral manifolds consist exclusively of tangency points.

Towards an estimate of the size of tangency sets, we need two technical lemmas, the first of which says, roughly speaking, that the differential of a form vanishes on the tangent set of the vanishing set of the form itself.

Lemma 3.1. *Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega \in \Omega^k(U)$ a k-form with C^1 coefficients on U. If $A \subseteq U$ and $\omega|_A \equiv 0$, then $(\mathrm{d}\omega)_x|_{\mathrm{Tan}_x(A)} = 0$ for all $x \in A$.*

Proof. We first prove the statement for zero-forms, i.e. for differentiable functions $f : U \to \mathbb{R}$, vanishing on A. Let $x \in A$. Without loss of generality, we may assume $\mathrm{Tan}_x(A) \neq \{0\}$. Consider $v \in \mathrm{Dir}_x(A)$ and a sequence (x_n) in A with $x_n \to x$, $x_n \neq x$, and $\frac{x_n - x}{|x_n - x|} \to v$. Using $f(x_n) = f(x) + (\mathrm{d}f)_x(x - x_n) + \mathrm{o}(|x - x_n|)$ and $f(x_n) = f(x) = 0$ for all $n \in \mathbb{N}$, it follows that

$$(\mathrm{d}f)_x(v) = \lim_{n\to\infty} (\mathrm{d}f)_x\left(\frac{x_n - x}{|x_n - x|}\right) = 0$$

and hence $(\mathrm{d}f)_x|_{\mathrm{Dir}_x(A)} = 0$. By the definition of $\mathrm{Tan}_x(A)$ and since $(\mathrm{d}f)_x$ is linear, it follows that $(\mathrm{d}f)_x|_{\mathrm{Tan}_x(A)} = 0$.

Now let us consider a k-form $\omega = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k}\, \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \in \Omega^k(U)$ with differential $\mathrm{d}\omega = \sum_{1\le i_1<\ldots<i_k\le n} \mathrm{d}a_{i_1\ldots i_k} \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$. Recall that the condition $\omega|_A \equiv 0$ is equivalent to $a_{i_1\ldots i_k}|_A \equiv 0$ for all indices $1 \le i_1 < \ldots i_k \le n$. Hence $\mathrm{d}a_{i_1\ldots i_k}|_{\mathrm{Tan}_x(A)} = 0$ for all $x \in A$ and for all indices $1 \le i_1 < \ldots < i_k \le n$. Now for $v_1, \ldots, v_{k+1} \in \mathrm{Tan}_x(A)$ and arbitrary indices $1 \le i_1 < \ldots < i_k \le n$ we have

$$(\mathrm{d}a_{i_1\ldots i_k} \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k})_x(v_1, \ldots, v_{k+1}) = 0,$$

which yields the claim. □

Lemma 3.2. *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$, $S \subseteq U$ a C^2 smooth n-dimensional manifold and let $\tau(S) = \tau(S, \mathcal{D})$ denote its tangency set with respect to $\mathcal{D}$. Then $(\mathrm{d}\vartheta^i)_z|_{\mathrm{Tan}_z(\tau(S))} = 0$ for all $1 \leq i \leq m$ and all $z \in \tau(S)$.*

Proof. Consider an open subset $V \subseteq \mathbb{R}^n$ and a C^2 mapping $f : V \to \mathbb{R}^m$ such that $\Phi : V \to \mathbb{R}^{n+m}$, $\Phi(x) = (x, f(x))$, is a local parametrisation of S. Note that $S = \Gamma_f$, where $\Gamma_f \subseteq \mathbb{R}^{n+m}$ denotes the graph of f and that $\Phi(x) \in \tau(\Gamma_f)$ if and only if $\vartheta^i_{\Phi(x)}|_{T_{\Phi(x)}(\Gamma_f)} = 0$ for all $1 \leq i \leq m$, which is equivalent to $(\Phi^*\vartheta^i)_x = 0$ for all $1 \leq i \leq m$. Since $\Phi \in C^2$ we have that the $\Phi^*\vartheta^i$ are one-forms on $\mathbb{R}^n$ with C^1 coefficients and Lemma 3.1 applies. Therefore $\Phi^*\vartheta^i|_{\tau(f)} = 0$ for all $1 \leq i \leq m$, where $\tau(f) := \Phi^{-1}(\tau(\Gamma_f))$, and $\mathrm{d}(\Phi^*\vartheta^i)_x|_{\mathrm{Tan}_x(\tau(f))} = (\Phi^* \,\mathrm{d}\vartheta^i)_x|_{\mathrm{Tan}_x(\tau(f))} = 0$ for all $1 \leq i \leq m$ and all $x \in \tau(f)$.

Now let $x \in \tau(f)$, which is equivalent to $z := \Phi(x) \in \tau(\Gamma_f) = \tau(S)$. By means of Proposition 2.2 3) we have $\mathrm{Tan}_z(\tau(S)) = \mathrm{Tan}_{\Phi(x)}(\Phi(\tau(f))) = (\mathrm{D}\Phi(x))(\mathrm{Tan}_x(\tau(f)))$ and therefore

$$(\Phi^* \,\mathrm{d}\vartheta^i)_x|_{\mathrm{Tan}_x(\tau(f))} = 0 \text{ if and only if } (\mathrm{d}\vartheta^i)_z|_{\mathrm{Tan}_z(\tau(S))} = 0$$

for all $1 \leq i \leq m$. □

Remark 3.3. Note that the C^2 property of S is necessary to apply Lemma 3.1 and it was also used in the equation $\mathrm{d}(\Phi^*\vartheta^i) = \Phi^* \,\mathrm{d}\vartheta^i$, which might not hold if Φ is less than C^2 smooth. If the assumption on C^2 smoothness is replaced by mere C^1 or even $C^{1,\alpha}$ smoothness, our results fail to hold, as we shall see in Section 8.

We observed that for a C^2 smooth manifold S and a C^1 smooth distribution $\mathcal{D}$ both the defining one-forms ϑ^i are zero on $\tau(S, \mathcal{D})$ as well as their differentials vanish on $\mathrm{Tan}_z(\tau(S))$ for $z \in \tau(S)$. This observation motivated the definition of the sets A_k from Definition 1.2.

After these preparations, we are now in position to prove our main result.

Proof of Theorem 1.3. Using the notation $\tau_k(S) := \tau(S) \cap (A_k \setminus A_{k+1})$ for $1 \leq k \leq n$, we decompose $\tau(S) = \tau(S, \mathcal{D})$ into suitable 'pieces':

$$\tau(S) = \tau(S) \cap [(A_1 \setminus A_2) \cup \ldots \cup (A_{n-1} \setminus A_n) \cup A_n] = \tau_1(S) \cup \ldots \cup \tau_n(S), \tag{3.4}$$

where $\tau_n(S) = \tau(S) \cap A_n$ since $A_{n+1} = \emptyset$. We first show that $\dim(\mathrm{Tan}_z(\tau_k(S))) \leq k$ for $1 \leq k \leq n$ and $z \in \tau_k(S)$: Let us assume by contradiction that there exist $1 \leq k \leq n$ and $z \in \tau_k(S)$ such that $\dim(\mathrm{Tan}_z(\tau_k(S))) \geq k + 1$. Then $\dim(\mathrm{Tan}_z(\tau(S)) \geq k + 1$ for $\mathrm{Tan}_z(\tau_k(S)) \preceq \mathrm{Tan}_z(\tau(S))$ by Proposition 2.2 1). Now choosing $X := \mathrm{Tan}_z(\tau(S))$ and using $(\mathrm{d}\vartheta^i)_z|_X = 0$ for $1 \leq i \leq m$ by Lemma 3.2 yields $z \in A_{k+1}$, which is a contradiction to the initial choice of z. Applying Proposition 2.7, it follows that $\dim_{\mathrm{H}}(\tau_k(S)) \leq k$ for $1 \leq k \leq n$. Therefore, as a consequence of the finite stability property [11] of the Hausdorff dimension, we get

$$\dim_{\mathrm{H}}(\tau(S)) \leq \max_{1 \leq k \leq n} \{\dim_{\mathrm{H}}(\tau_k(S))\},$$

which implies the claim. □

Remark 3.4. We point out that an even stronger version of our main theorem holds: The range of k at the right side of (1.2) can be restricted to $\lfloor \frac{n+m}{m+1} \rfloor \leq k \leq n$ as a consequence of the following Proposition. By $\lfloor r \rfloor$, $r \in \mathbb{R}$, we denote the integer part of r, i.e. the greatest integer such that $\lfloor r \rfloor \leq r$.

Proposition 3.5. *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and A_k, $k \in \mathbb{N}$, defined as in Definition 1.2. Then $A_k = U$ for all $1 \leq k \leq \lfloor \frac{n+m}{m+1} \rfloor$.*

Proof. We shall show the assertion by induction on k. Clearly, $A_1 = U$ by the integrability of vector fields [2]. Now let us assume that $A_k = U$ for some $1 \le k \le \lfloor \frac{n-1}{m+1} \rfloor$ and consider $z \in U$. Then there exists $X \in G(n+m,k)$ such that $\vartheta^i_z|_X = 0$ and $\mathrm{d}\vartheta^i_z|_X = 0$ for $1 \le i \le m$. Consider a basis $\mathcal{B} = \{e_1, \ldots, e_k\}$ of X and let $\mathcal{B}' = \{e'_{k+1}, \ldots, e'_n\}$ be a completion of $\mathcal{B}$ to a basis of $\mathcal{D}(z)$.

In order to show $z \in A_{k+1}$, we shall find $e_{k+1} = a_{k+1} \cdot e'_{k+1} + \ldots + a_n \cdot e'_n \in \mathcal{D}(z)$, such that

$$\mathrm{d}\vartheta^i(e_l, e_{k+1}) = 0 \tag{3.5}$$

for all $1 \le l \le k$ and $1 \le i \le m$. First note that e_{k+1} is only determined up to its length. Therefore, we may assume without loss of generality that $a_{k+1} = 1$. Hence, condition (3.5) yields a system of $k \cdot m$ equations with $n - k - 1$ unknowns, which is solvable, since $k \le \lfloor \frac{n-1}{m+1} \rfloor$ is equivalent to $k \cdot m \le n - k - 1$. Thus, $X' := \mathrm{span}\{e_1, \ldots, e_{k+1}\}$ is a $(k+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$ such that $\mathrm{d}\vartheta_z|_{X'} = 0$, which immediately implies that $z \in A_{k+1}$. □

4. Generalisation of Derridj's Theorem

As a first application of our main theorem in the context of Hörmander distributions [15], we are going to present a generalisation of Derridj's Theorem [9] on the size of tangencies. In this direction, we shall start with the following

Definition 4.1. Let $\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_n\}$ be a C^∞ smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$. We say that $\mathcal{D}$ fulfils the Hörmander condition if for any point $z \in U$, the vectors generated by the iterated brackets

$$X_{i_1}(z), [X_{i_1}, X_{i_2}](z), [X_{i_1}, [X_{i_2}, X_{i_3}]](z), \ldots,$$

where $1 \le i_k \le n$ for any $k \in \mathbb{N}$, span the whole space $T_z(\mathbb{R}^{n+m})$.

Remark 4.2. Even though not very often used in literature, one may extend the notion of being Hörmander to merely C^r, $r \in \mathbb{N}$, distributions. Since in this case the interlaced brackets are only defined up to order $r+1$, one has to require already the latter to span $T_z(\mathbb{R}^{n+m})$ for all $z \in U$.

Before stating the main theorem of this section, we shall state and prove the following proposition on the size of tangencies in the context of a *Hörmander* distribution. We shall also generalise the notion of the tangency set to the needs of the respective theorem.

Proposition 4.3. *Let $\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_n\} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^∞ smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ that fulfils the Hörmander condition, $S \subseteq U$ a C^2 smooth n-dimensional manifold and $\tau(S) = \tau(S, \mathcal{D})$ its tangency set with respect to $\mathcal{D}$. Then $\dim_{\mathrm{H}}(\tau(S)) \le n - 1$.*

Proof. Using the decomposition of $\tau(S)$ according to (3.4), it is enough to show that $\dim_{\mathrm{H}}(\tau_n(S)) \le n - 1$, which we shall do by 'stratifying' $\tau_n(S)$ according to the pointwise 'degree of involutivity': Define, for $k \in \mathbb{N}$, the sets

$$\begin{aligned} B_k := \{z \in \tau_n(S) \quad : \quad & [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{j-1}}, X_{i_j}] \ldots]](z) \in \mathcal{D}(z) \\ & \text{for all } 1 \le j \le k \text{ and all } 1 \le i_1, \ldots, i_j \le n\} \end{aligned}$$

and note that the B_k form a decreasing sequence in the sense $\tau_n(S) = B_1 \supseteq B_2 \supseteq \ldots$. Therefore, $\tau_n(S)$ can be decomposed as

$$\tau_n(S) = \bigcup_{k=1}^{\infty} (B_k \setminus B_{k+1}) \cup \bigcap_{k=1}^{\infty} B_k.$$

Since $\bigcap_{k=1}^{\infty} B_k = \emptyset$ by the Hörmander condition on $\mathcal{D}$ and using the countable stability property [11] of the Hausdorff dimension, it is thus enough to show $\dim_{\mathrm{H}}(B_k) \le n-1$ for all $k \in \mathbb{N}$.

Let us assume that there exists $k \in \mathbb{N}$ such that $\dim_{\mathrm{H}}(B_k \setminus B_{k+1}) > n-1$. Then there exists $z \in B_k \setminus B_{k+1}$ such that $\dim(\mathrm{Tan}_z(B_k \setminus B_{k+1})) = n$ by Proposition 2.7. Applying Proposition 2.2 1) to the inclusion $B_k \setminus B_{k+1} \subseteq S$ and taking into account their respective dimensions, we get

$$\mathrm{Tan}_z(B_k \setminus B_{k+1}) = \mathrm{Tan}_z(S) = T_z(S). \tag{4.1}$$

Next consider the multi-index $\mathbf{i} := (i_1, \ldots, i_k)$, where $1 \le i_1, \ldots, i_k \le n$, and denote by $X_{\mathbf{i}}$ the vector field $[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]$. Note that $X_{\mathbf{i}}(z) \in \mathcal{D}(z)$, which is equivalent to $\vartheta_z^i(X_{\mathbf{i}}(z)) = 0$ for all $1 \le i \le m$. Also consider any $1 \le i_0 \le n$ and the vector field $X_{i_0} \in \mathcal{D}$. Using $\vartheta^i(X_{i_0}) \equiv 0$ for all $1 \le i \le m$ and $z \in A_n$, we get

$$\begin{aligned} \vartheta_z^i([X_{i_0}, X_{\mathbf{i}}]) &= X_{i_0}(z)(\vartheta_z^i(X_{\mathbf{i}})) - X_{\mathbf{i}}(z)(\vartheta_z^i(X_{i_0})) - (\mathrm{d}\vartheta^i)_z(X_{i_0}(z), X_{\mathbf{i}}(z)) \\ &= X_{i_0}(z)(\vartheta^i(X_{\mathbf{i}})) = \mathrm{d}(\vartheta^i(X_{\mathbf{i}}))_z(X_{i_0}(z)) \end{aligned}$$

for all $1 \le i \le m$. Next observe that $\vartheta^i(X_{\mathbf{i}})|_{B_k \setminus B_{k+1}} \equiv 0$ since $X_{\mathbf{i}}(z') \in \mathcal{D}(z')$ for all $z' \in B_k \setminus B_{k+1} \subseteq \tau_n(S)$. According to Lemma 3.1 and using (4.1), this yields $\mathrm{d}(\vartheta^i(X_{\mathbf{i}}))_z|_{T_z(S)} = 0$ for $1 \le i \le m$ and therewith $\mathrm{d}(\vartheta^i(X_{\mathbf{i}}))_z(X_{i_0}(z)) = 0$ for $1 \le i \le m$. Hence $\vartheta_z^i([X_{i_0}, X_{\mathbf{i}}](z)) = 0$ for all $1 \le i \le n$ and thus

$$[X_{i_0}, X_{\mathbf{i}}] = [X_{i_0}, [X_{i_1}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]] \in \mathcal{D}(z).$$

But this implies, since both i_0 and $\mathbf{i}$ may be chosen arbitrarily, $z \in B_{k+1}$, which is contradictory to the initial choice of z. □

Definition 4.4. Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and $S \subseteq U$ a C^1 smooth $(n+r)$-dimensional manifold, where $0 \le r \le m$. Then we define

$$\tilde{\tau}(S, \mathcal{D}) := \{z \in S : T_z(S) \supseteq \mathcal{D}(z)\}.$$

Note that for $r = 0$ we have $\tilde{\tau}(S, \mathcal{D}) = \tau(S, \mathcal{D})$ and that the following theorem reduces to Proposition 4.3 in this case.

Theorem 4.5. *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m) = \mathrm{span}\{X_1, \ldots, X_n\}$ be a C^∞ smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ that fulfils the Hörmander condition. Let further $S \subseteq U$ be an $(n+r)$-dimensional C^2 smooth manifold, where $0 \le r < m$, and $\tilde{\tau}(S, \mathcal{D})$ its tangency set with respect to $\mathcal{D}$. Then*

$$\dim_{\mathrm{H}}(\tilde{\tau}(S, \mathcal{D})) \le n + r - 1.$$

Proof. The idea of the proof is to reduce the statement to Proposition 4.3. To do so let $\mathcal{F} = \{Y_1, \ldots, Y_{n+r}\}$ be a C^2 smooth frame of $T(S)$, i.e. a C^2 smooth distribution of rank $n+r$ on S such that $\mathrm{span}\{Y_1(z), \ldots, Y_{n+r}(z)\} = T_z(S)$ for all $z \in S$. Then there exist a neighbourhood U' of S and an extension $\hat{\mathcal{F}} = \{\hat{Y}_1, \ldots, \hat{Y}_{n+r}\}$ of $\mathcal{F}$ on U', i.e. $\hat{\mathcal{F}}|_S = \mathcal{F}$, such that $\hat{\mathcal{F}}(z) := \{\hat{Y}_1(z), \ldots, \hat{Y}_{n+r}(z)\}$ is linearly independent for all $z \in U'$. Without loss of generality, we may assume that $U' = U$.

We will next complete $\mathcal{D}$ to a C^1 smooth distribution $\tilde{\mathcal{D}}$ of rank $n+r$ by adding r vector fields of $\hat{\mathcal{F}}$ to $\mathcal{D}$ using the following iterative construction: For $1 \le k \le n+r$ and fixed $z_0 \in U$ define $\mathcal{D}_k(z_0) := \mathrm{span}\{\mathcal{D}_{k-1}(z_0), \hat{Y}_k(z_0)\}$, where $\mathcal{D}_0 := \mathcal{D}$. Note that if $\hat{Y}_k(z_0) \notin \mathcal{D}_{k-1}(z_0)$ there exists a neighbourhood U' of z_0 such that $\hat{Y}_k(z) \notin \mathcal{D}_{k-1}(z)$ for all $z \in U'$, since $\mathcal{D}_{k-1}$ and $\hat{Y}_k$ are continuous. We may again assume that $U' = U$ for otherwise we can cover S by countably many such neighbourhoods. Hence the construction of the $\mathcal{D}_k$ does not depend on U, which implies that the $\mathcal{D}_k$ are actually C^2 smooth distributions on U. Observe that there exists a (least) k_0, $r \le k_0 \le n+r$, such that $\mathrm{rank}(\mathcal{D}_{k_0}) = n+r$.

Define the distribution $\tilde{\mathcal{D}} := \mathcal{D}_{k_0} = \operatorname{span}\{\mathcal{D}, \hat{Y}_{k_1}, \ldots, \hat{Y}_{k_r}\}$, $1 \leq k_1 < \ldots < k_r \leq n + r$, and observe that $\tilde{\mathcal{D}}$ fulfils the Hörmander condition.

The final step towards the completion of the proof is the observance that the equality $\tilde{\tau}(S, \mathcal{D}) = \tau(S, \tilde{\mathcal{D}})$ holds: For the first inclusion consider $z \in \tilde{\tau}(S, \mathcal{D})$. Then $T_z(S) \supseteq \mathcal{D}(z)$ and, since $\operatorname{span}\{\hat{Y}_{k_1}(z), \ldots, \hat{Y}_{k_r}(z)\} \subseteq T_z(S)$, we get $T_z(S) = \tilde{\mathcal{D}}(z)$. For the opposite inclusion let $z \in \tau(S, \tilde{\mathcal{D}})$. Then $T_z(S) = \tilde{\mathcal{D}}(z)$ and, since $\mathcal{D}(z) \subseteq \tilde{\mathcal{D}}(z)$, $T_z(S) \supseteq \mathcal{D}(z)$. Now the claim follows immediately from Proposition 4.3, applied to $\tilde{\mathcal{D}}$ and S. $\square$

The following theorem of Derridj [9, Theorem 1] is an immediate consequence of Theorem 4.5.

Corollary 4.6 (Derridj's Theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with regular, i.e. C^∞ smooth, boundary and let $\mathcal{D} = \{X_1, \ldots, X_r\}$ be a system of vector fields with C^∞ coefficients on a neighbourhood Ω' of $\overline{\Omega}$ that fulfils the Hörmander condition. Then*

$$\mathcal{H}^{n-1}(\tilde{\tau}(\partial\Omega, \mathcal{D})) = 0,$$

where $\tilde{\tau}(\partial\Omega, \mathcal{D})$ denotes the (closed) tangency set of the boundary $\partial\Omega$ of Ω with respect to $\mathcal{D}$.

Remark 4.7. Note that our Theorem 4.5 is stronger than Derridj's Theorem in at least three directions: First, Theorem 4.5 does not only hold for one-codimensional manifolds or boundaries respectively. Secondly, our theorem estimates the size of the tangency set not only in terms of its Hausdorff *measure*, but rather with regard to its Hausdorff *dimension*. And thirdly, our version is not restricted to *bounded* surfaces.

5. Lower estimates

In the sequel, we are going to provide a *lower bound* for the maximal size of the tangency set with respect to a given distribution. Since this lower bound coincides surprisingly often with the *upper bound* from Theorem 1.3, the latter is in fact sharp in many cases. We are going to provide examples featuring this coincidence in Section 6. A useful notion to get a lower estimate is introduced in the following

Definition 5.1. Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and $z \in U$. Then we define the *involutivity index of $\mathcal{D}$ at z* as

$$\iota_z(\mathcal{D}) := \max\{\dim(S') : S' \in \mathcal{S}(z) \text{ and } T_{z'}(S') \subseteq \mathcal{D}(z') \text{ for all } z' \in S'\},$$

where $\mathcal{S}(z)$ denotes the collection of all C^2 smooth submanifolds of U passing through z. Furthermore, we define the *involutivity index of $\mathcal{D}$* as

$$\iota(\mathcal{D}) := \max\{\iota_z(\mathcal{D}) : z \in U\}.$$

Combining Theorem 1.3, or rather its stronger version according to Remark 3.4, with the involutivity index, we get the following *two-sided* estimate for the size of tangencies.

Theorem 5.2. *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$, $\iota(\mathcal{D})$ its involutivity index and A_k defined as in Definition 1.2. Then there exists a C^2 smooth n-dimensional manifold $S \subseteq U$ such that*

$$\iota(\mathcal{D}) \leq \dim_{\mathrm{H}}(\tau(S, \mathcal{D})) \leq \max_{\lfloor \frac{n+m}{m+1} \rfloor \leq k \leq n} \{\min\{\dim_{\mathrm{H}}(A_k \setminus A_{k+1}), k\}\}. \tag{5.1}$$

Proof. Theorem 1.3 already asserts the second inequality for any C^2 smooth n-dimensional manifold $S \subseteq U$. It is therefore enough to construct a manifold $S \subseteq U$ with the required properties that fulfils the left inequality. In this direction, let $z \in U$ such that $k := \iota_z(\mathcal{D}) = \iota(\mathcal{D})$ and let $S' \in \mathcal{S}(z)$ be a C^2 smooth k-dimensional manifold passing through z such that $T(S') \subseteq \mathcal{D}$ (which exists by the definition of $\iota_z(\mathcal{D})$). Consider k pointwise linearly independent vector fields $\{Y_1, \ldots, Y_k\}$ on S' that span the local subdistribution $T(S') \subseteq \mathcal{D}$, i.e. such that $\operatorname{span}\{Y_1(v), \ldots, Y_k(v)\} = T_v(S')$ holds for all $v \in S'$. Denote by $\{\varphi_1(t), \ldots, \varphi_k(t)\}$ the local flows of $\{Y_1, \ldots, Y_k\}$ at z and recall that $\varphi_i(t) \circ \varphi_j(s) = \varphi_j(s) \circ \varphi_i(t)$ for $1 \leq i, j \leq k$, whenever the composed diffeomorphisms are defined.

We next extend $\{Y_1, \ldots, Y_k\}$ to a system $\{Y_1, \ldots, Y_k, Y_{k+1}, \ldots, Y_n\}$ that spans the whole subbundle $\mathcal{D}|_{S'}$. Observe that, for a suitable choice of ε, $\varphi : (-\varepsilon, \varepsilon)^k \to U$, $\varphi(t_1, \ldots, t_k) = (\varphi_1(t_1) \circ \ldots \circ \varphi_k(t_k))(z)$, is a local parametrisation of S' and the mapping $\psi : (-\varepsilon, \varepsilon)^n \to U$, defined by

$$\psi(t_1, \ldots, t_n) := t_{k+1} \cdot Y_{k+1}(\varphi(t_1, \ldots, t_k)) + \ldots + t_n \cdot Y_n(\varphi(t_1, \ldots, t_k)),$$

is an embedding into U for which $T_v(\operatorname{Im}(\psi)) = \mathcal{D}(v)$ for all $v \in \operatorname{Im}(\varphi)$. This shows that $\operatorname{Im}(\varphi) \subseteq \tau(\operatorname{Im}(\psi), \mathcal{D})$ and hence $\iota_z(\mathcal{D}) \leq \dim_{\mathrm{H}}(\tau(\operatorname{Im}(\psi), \mathcal{D}))$. Since $S := \operatorname{Im}(\psi)$ is a C^2 smooth manifold in U and $\iota_z(\mathcal{D}) = \iota(\mathcal{D})$, the claim follows. □

Interestingly, an arbitrary one-codimensional distribution $\mathcal{D}$ contains, at least on an open subset, an involutive subdistribution $\mathcal{D}'$ of rank roughly equal to half the dimension of the ambient space. The rank of $\mathcal{D}'$ can then be used as a lower bound for $\iota(\mathcal{D})$.

Before stating and proving the respective theorem, we shall define the notion of the *rank of a two-form* as follows.

Definition 5.3. Let ω be a C^1 smooth two-form on an open set $U \subseteq \mathbb{R}^{n+m}$. Then the rank of ω is

$$\operatorname{rank}(\omega) := \min\{i \in \mathbb{N} : (\mathrm{d}\omega)^i \equiv 0 \text{ on } U\}.$$

Note that the rank of a two-form may decrease when restricting U. However, it is lower semicontinuous on U.

Theorem 5.4. *Let $\mathcal{D} = \ker(\vartheta)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+1}$. Then there exists an open set $U' \subseteq U$ and an involutive subdistribution $\mathcal{D}'$ of $\mathcal{D}$ on U' of $\operatorname{rank}(\mathcal{D}') = \lfloor \frac{n+1}{2} \rfloor$.*

Proof. We shall show the existence of $\mathcal{D}'$ using Sternberg's version of Darboux' Theorem [24]. We point out that [24] assumes C^∞ smoothness of $\mathcal{D}$. However, the assertion still holds for mere C^1 smoothness, as the reader may verify. Let $p := \operatorname{rank}(\mathrm{d}\vartheta)$ and consider $z \in U$ such that $(\mathrm{d}\vartheta)^p_z \neq 0$. Observe that $(\mathrm{d}\vartheta)^p \neq 0$ on a whole neighbourhood of z.

Let us first assume that there exists a neighbourhood $U' \subseteq U$ of z such that $(\mathrm{d}\vartheta)^p \neq 0$ but $\vartheta \wedge (\mathrm{d}\vartheta)^p \equiv 0$ on U'. Hence, according to Darboux' Theorem, there exist coordinates $x_1, \ldots, x_p, y_1, \ldots, y_p, x_{p+1}, \ldots, x_{n+1-p}$ on U' such that ϑ is diffeomorphic to the standard form

$$\vartheta_0 := \sum_{i=1}^{p} x_i \cdot \mathrm{d}y_i$$

on U'. Observe that

$$\mathcal{D}_0' := \operatorname{span}\{\partial_{x_1}, \ldots, \partial_{x_{n+1-p}}\}$$

is an involutive subdistribution of $\mathcal{D}_0 := \ker(\vartheta_0)$ of rank $n + 1 - p$. Also note that, since $(\mathrm{d}\vartheta)^p$ is a non-vanishing $2p$-form on U', we have $p \leq \lfloor \frac{n+1}{2} \rfloor$. Since the involutivity property of a distribution is preserved under diffeomorphisms, this yields the existence of the desired subdistribution.

If, on the other hand, there is no neighbourhood of z on which $\vartheta \wedge (\mathrm{d}\vartheta)^p \equiv 0$, there exist $z' \in U$ and a neighbourhood $U' \subseteq U$ of z' on which $\vartheta \wedge (\mathrm{d}\vartheta)^p \neq 0$. Now using the same arguments as above with the standard form $\vartheta_0 := \sum_{i=1}^{p} x_i \cdot \mathrm{d}y_i + x_{p+1}$, we get the involutive subdistribution $\mathcal{D}_0' := \operatorname{span}\{\partial_{x_1}, \ldots, \partial_{x_p}, \partial_{x_{p+2}}, \ldots, \partial_{x_{n+1-p}}\}$ of rank $n - p$. Since in this case $p \leq \lfloor \frac{n}{2} \rfloor$, the existence of $\mathcal{D}'$ follows likewise. □

As an immediate consequence of Theorem 5.4 in combination with Frobenius' Theorem [2, 20, 24], we get

Corollary 5.5. *Let $\mathcal{D} = \ker(\vartheta)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+1}$ and $\iota(\mathcal{D})$ its involutivity index. Then*

$$\iota(\mathcal{D}) \geq \lfloor \tfrac{n+1}{2} \rfloor.$$

Remark 5.6. Theorem 5.4 shows that it makes sense to take $k \geq \lfloor \frac{n+m}{m+1} \rfloor$ in the statement of Theorem 5.2 in the case when $m = 1$. We do not know, however, whether a similar statement to Theorem 5.4 holds for arbitrary $m \in \mathbb{N}$.

6. Applications to contact and symplectic structures

Reviving the spirit of Section 4, we are going to illustrate the usefulness of our main Theorem 1.3 in terms of more specific applications in this section. Attempting to provide a wide variety of interesting and important examples, we shall investigate the cases of contact and symplectic structures along with distributions on Carnot groups, where the latter will be represented by the Heisenberg and the Engel groups. We will be able to recapture a result by the first author [3] in an elegant and straightforward way. For technical reasons, the notion of *vanishing sets*, earlier used by the second author [22], turns out to be useful in this context.

We start with the following

Definition 6.1. Let $U \subseteq \mathbb{R}^{n+m}$ be an open set and let η be a k-form on U. Then we define the *vanishing set of* η, i.e. the collection of points $z \in U$ at which η is (identically) zero as

$$V(\eta) := \{z \in U : \eta_z(v_1, \ldots, v_k) = 0 \text{ for all } v_i \in T_z(\mathbb{R}^{n+m})\}.$$

Let further $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^0 distribution of rank n on U. Then, for $s \in \mathbb{N}_0$, we define the *s-vanishing set*, or just *vanishing set*, with respect to $\mathcal{D}$ as

$$V_s := \bigcap_{i_1,\ldots,i_s=1}^{m} V(\vartheta^1 \wedge \ldots \wedge \vartheta^m \wedge \mathrm{d}\vartheta^{i_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{i_s}).$$

Remark 6.2. Let $\mathcal{D}$ be a C^0 distribution as defined above and V_s, $s \in \mathbb{N}_0$, the corresponding vanishing sets. Then the following holds, as may easily be verified.

1) V_s is a closed subset of U for any $s \in \mathbb{N}_0$.
2) We have $V_0 = \emptyset$ and $V_s = U$, whenever $s > \frac{n}{2}$.
3) The vanishing sets V_s, $1 \leq s \leq \lfloor \frac{n+1}{2} \rfloor$, form a monotonic sequence in the sense that $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{\lfloor \frac{n-1}{2} \rfloor} \subseteq V_{\lfloor \frac{n+1}{2} \rfloor} = U$.
4) $\mathcal{D}$ is involutive if and only if $V_1 = U$; see above or [2, Theorem 3.8].

The usefulness of the vanishing sets for our purposes is founded in their relationship with the sets A_k as follows.

Proposition 6.3. 1) *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and A_k, $1 \leq k \leq n+m$, as well as V_s, $s \in \mathbb{N}_0$, be defined as above. Then*

$$A_{n-s+1} \subseteq V_s$$

for all $1 \leq s \leq n$.

2) *Let furthermore $S \subseteq U$ be an n-dimensional C^2 smooth manifold on U and let $\tau(S) = \tau(S, \mathcal{D})$ denote its tangency set with respect to $\mathcal{D}$. Then*

$$\dim_{\mathrm{H}}(\tau(S \setminus V_s)) \leq n - s$$

for all $1 \leq s \leq n$. In particular $\dim_{\mathrm{H}}(\tau(S)) \leq n - s$ for any $S \subseteq U \setminus V_s$.

Note that both assertions are trivially fulfilled for $\lfloor \frac{n+1}{2} \rfloor \le s \le n$.

Proof. 1) Consider $z \in A_{n-s+1}$ and $X \in G(n+m, n-s+1)$ such that $\vartheta^i_z|_X = 0$ and $\mathrm{d}\vartheta^i_z|_X = 0$ for all $1 \le i \le m$. Let $\mathcal{B} = \{e_1, \ldots, e_{n-s+1}\}$ denote a basis of X and let $\mathcal{B}' = \{e_{n-s+2}, \ldots, e_{n+m}\}$ be a completion of $\mathcal{B}$ to a basis of $\mathbb{R}^{n+m}$. Observe that $\vartheta^1 \wedge \ldots \wedge \vartheta^m \wedge \mathrm{d}\vartheta^{i_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{i_s}(e_{l_1}, \ldots, e_{l_{m+2s}})$ is a sum of products of type

$$\pm\vartheta^1(e_{l_1})\cdot\ldots\cdot\vartheta^m(e_{l_m})\cdot\mathrm{d}\vartheta^{i_1}(e_{l_1}, \ldots, e_{l_{m+2s}})\cdot\ldots\cdot\mathrm{d}\vartheta^{i_s}(e_{l_{m+2s-1}}, e_{l_{m+2s}}), \tag{6.1}$$

where $(l_1, \ldots, l_{m+2s})$ is a permutation of $(1, \ldots, m+2s)$. From $\mathrm{card}(\mathcal{B}'_X) = m+s-1$ it follows that any product of type (6.1) contains at least $s+1$ arguments from $\mathcal{B}_X$. Hence, at least one factor ϑ^i or $\mathrm{d}\vartheta^i$ has arguments exclusively from X and is zero therefore. Thus $z \in V_s$.

2) Without loss of generality, we may assume that $S \subseteq U \setminus V_s$. Using the notations from the proof of Theorem 1.3, we observe that the inclusions

$$\tau_k(S) = \tau(S) \cap (A_k \setminus A_{k+1}) \subseteq A_k \subseteq A_{n-s+1} \subseteq V_s$$

hold for any $n-s+1 \le k \le n+m$. Together with the inclusion $\tau_k(S) \subseteq S \subseteq U \setminus V_s$, this implies that $\tau_k(S) \subseteq V_s \cap (U \setminus V_s) = \emptyset$ for any $n-s+1 \le k \le n+m$. Therefore $\tau(S) = \tau_1(S) \cup \ldots \cup \tau_{n-s}(S)$.

Now the inequalities $\dim_{\mathrm{H}}(\tau_k(S)) \le k$ for $1 \le k \le n-s$ from the proof of Theorem 1.3 together with the finite stability property [11] of the Hausdorff dimension imply the claim. ☐

We are now able to apply our considerations in the following examples.

Example 6.4. Let ξ be a contact structure on an open set $U \subseteq \mathbb{R}^{2n+1}$, i.e. a distribution $\mathcal{D}$ of codimension one that locally is the kernel of a contact form. Recall that a contact form η on U is a one-form such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on U, which means that $V_n = \emptyset$. Therefore $A_{n+1} = \emptyset$ by Proposition 6.3 1), such that Theorem 1.3 implies that $\dim_{\mathrm{H}}(\tau(S, \mathcal{D})) \le n$ for any C^2 smooth hypersurface $S \subseteq U$.

Example 6.5. Consider the Heisenberg group $\mathbb{H}^n$ with underlying space $\mathbb{R}^{2n+1}$ and its points denoted by $z = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ along with the so-called horizontal distribution $\mathcal{H}$, generated by the $2n$ pointwise linearly independent vector fields

$$X_i = \partial_{x_i} + 2y_i\partial_t \qquad \text{and} \qquad Y_i = \partial_{y_i} - 2x_i\partial_t, \tag{6.2}$$

where $1 \le i \le n$. The reader may easily verify that $\mathcal{H}$ fulfils the Hörmander condition and that the one-form corresponding to $\mathcal{H}$ is, according to (3.2),

$$\vartheta = 2\sum_{i=1}^{n}(y_i\,\mathrm{d}x_i - x_i\,\mathrm{d}y_i) - \mathrm{d}t \tag{6.3}$$

and is a contact form on $\mathbb{H}^n$. Now by the considerations in Example 6.4, we can immediately recapture the result [3, Theorem 1.2] of the first author stating that

$$\dim_{\mathrm{H}}(\tau(S, \mathcal{H})) \le n$$

for any C^2 smooth hypersurface $S \subseteq \mathbb{H}$.

It turns out that the notion of vanishing sets introduced in Definition 6.1 might be inappropriate under certain circumstances. Examples include e.g. odd-codimensional distributions on even-dimensional ambient spaces, such as symplectic structures or hyperdistributions on the Engel group.

We are therefore going to introduce a more general notion of vanishing sets, allowing wedge products of *any* number of one-forms, as follows.

Definition 6.6. Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^0 distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$. Then, for $1 \leq r \leq m$ and $s \in \mathbb{N}_0$, we define the (r,s)*-vanishing set*, or just *vanishing set*, with respect to $\mathcal{D}$ as

$$V_{r,s} := \bigcap_{\substack{1 \leq i_1 < \ldots < i_r \leq m \\ 1 \leq j_1 \leq \ldots \leq j_s \leq m}} V(\vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_r} \wedge \mathrm{d}\vartheta^{j_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{j_s}).$$

Remark 6.7. This more general notion of vanishing sets has similar properties to the original one. Mentionable are, amongst others,

1) $V_{m,s} = V_s$ for any $s \in \mathbb{N}_0$ and $V_{r,0} = \emptyset$ for any $1 \leq r \leq m$, where the latter is due to the linear independence of the forms ϑ^i, $1 \leq i \leq m$.
2) $V_{r,s} \subseteq V_{r',s'}$ whenever $(r,s) \leq (r',s')$ and $V_{r,s} = U$ whenever $r + 2s > n + m$.
3) The inclusion $A_{n+m-r-s+1} \subseteq V_{r,s}$ holds for all $r, s \in \mathbb{N}_0$ such that $r + 2s \leq n + m$. This can be shown exactly the same way as in the proof of Proposition 6.3 1).

Using this more general definition of the vanishing sets, we get another corollary of our main theorem.

Corollary 6.8. *Let $\mathcal{D} = \ker(\vartheta^1) \cap \ldots \cap \ker(\vartheta^m)$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and define*

$$\Omega := \{\omega = \vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_r} \wedge \mathrm{d}\vartheta^{j_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{j_s} \wedge \zeta : \omega \text{ volume form on } U\}, \tag{6.4}$$

where $1 \leq i_1 < \ldots < i_r \leq m$, $1 \leq j_1 \leq \ldots\ldots \leq j_s \leq m$ and ζ is an $(n+m-r-2s)$-form on U. Let further $S \subseteq U$ be a C^2 smooth n-dimensional manifold and $\tau(S,\mathcal{D})$ its tangency set with respect to $\mathcal{D}$. Then

$$\dim_{\mathrm{H}}(\tau(S,\mathcal{D}) \leq \min_{\Omega}\{n + m - r - s\} \leq n.$$

Proof. First note that $\Omega \neq \emptyset$, since there always exists an n-form ζ on U such that $\omega = \bigwedge_{i=1}^m \vartheta^i \wedge \zeta$ is a volume form on U. This already implies the second inequality. For the first inequality, let $r_0, s_0 \in \mathbb{N}_0$ such that $n + m - r_0 - s_0 = \min_\Omega\{n + m - r - s\}$ and let $\omega_0 = \vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_{r_0}} \wedge \mathrm{d}\vartheta^{j_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{j_{s_0}} \wedge \zeta_0 \in \Omega$ be a volume form on U that realises this minimum. Observe that $V(\vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_{r_0}} \wedge \mathrm{d}\vartheta^{j_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{j_{s_0}}) = \emptyset$ in order ω_0 being a volume form on U. Now applying Remark 6.7 3), we get

$$A_{n+m-r_0-s_0+1} \subseteq V_{r_0,s_0} \subseteq V(\vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_{r_0}} \wedge \mathrm{d}\vartheta^{j_1} \wedge \ldots \wedge \mathrm{d}\vartheta^{j_{s_0}}) = \emptyset, \tag{6.5}$$

such that the claim follows from Theorem 1.3. $\square$

Examples applying the above Corollary include

Example 6.9. Consider the (C^∞ smooth) one-form $\vartheta := x_1 \mathrm{d}y_1 + \ldots + x_n \mathrm{d}y_n$ on the open set $U := (0,1)^{2n}$ and the corresponding distribution $\mathcal{D} = \ker(\vartheta)$ of rank $2n-1$. Observe that the differential $\mathrm{d}\vartheta = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$ is the standard symplectic form on U and that $(\mathrm{d}\vartheta)^n$ is a volume form on U. Now using Corollary 6.8, we immediately get $\dim_{\mathrm{H}}(\tau(S,\mathcal{D})) \leq n$ for any C^2 smooth hypersurface $S \subseteq U$.

A generalisation of Example 6.5 to a higher codimensional situation reads as follows.

Example 6.10. Consider the Heisenberg group $\mathbb{H}^n$ and its standard horizontal vector fields X_i, Y_i defined as in (6.2). Let

$$\mathcal{D} = \mathrm{span}\{X_{i_1}, \ldots, X_{i_p}, Y_{j_1}, \ldots, Y_{j_s}\}$$

be the distribution pointwise spanned by $0 \leq p \leq n$ vector fields X_i and $0 \leq s \leq n$ vector fields Y_j. Let further denote $S \subseteq \mathbb{H}^n$ a C^2 smooth $(k := p+s)$-dimensional manifold and $\tau(S,\mathcal{D})$ its tangency set with respect to $\mathcal{D}$. Then

$$\dim_{\mathrm{H}}(\tau(S,\mathcal{D})) \leq \mathrm{card}(I \cup J) = p + s - \mathrm{card}(I \cap J), \tag{6.6}$$

where $I := \{i_1, \ldots, i_p\}$ and $J := \{j_1, \ldots, j_s\}$: Indeed, without loss of generality, we may assume that $\mathcal{D} = \{X_1, \ldots, X_p, Y_{q+1}, \ldots, Y_r\}$, where $0 \le q \le p \le r \le n$. In this case, $k = p + r - q$, $I = \{1, \ldots, p\}$, $J = \{q+1, \ldots, r\}$ and $r = p + s - \operatorname{card}(I \cap J)$.

Note that $\mathcal{D}$ is, according to (3.2) and letting $N := \{1, \ldots, n\}$, the intersection of the kernels of the one-forms $\vartheta^i = -\,\mathrm{d}x_i$ for $i \in N \setminus I$, $\eta^j = -\,\mathrm{d}y_j$ for $j \in N \setminus J$ and $\xi = 2\sum_{i\in I} y_i\,\mathrm{d}x_i - 2\sum_{j\in J} x_j\,\mathrm{d}y_j - \mathrm{d}t$,

$$\mathcal{D} = \bigcap_{i\in N\setminus I} \ker(\vartheta^i) \cap \bigcap_{j\in N\setminus J} \ker(\eta^j) \cap \ker(\xi).$$

Note further that $\mathrm{d}\vartheta^i = 0$ for $i \in N \setminus I$, $\mathrm{d}\eta^j = 0$ for $j \in N \setminus J$ and

$$\mathrm{d}\xi = -2 \sum_{i\in I\triangle J} \mathrm{d}x_i \wedge \mathrm{d}y_i - 4 \sum_{i\in I\cap J} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Hence $(\mathrm{d}\xi)^r = c\bigwedge_{i=1}^r(\mathrm{d}x_i \wedge \mathrm{d}y_i)$, where $c \neq 0$ depends on p, s and r, and therefore

$$\omega := \bigwedge_{i=r+1}^{n} (\vartheta^i \wedge \eta^i) \wedge \xi \wedge (\mathrm{d}\xi)^r = c \bigwedge_{i=1}^{n} (\mathrm{d}x_i \wedge \mathrm{d}y_i) \wedge \mathrm{d}t$$

is a volume form on $\mathbb{H}^n$. Now (6.6) follows from Corollary 6.8.

In all the above examples, it was possible to obtain a volume form as a wedge product exclusively made up by the defining one-forms and their differentials. It is therefore a natural question whether the form ζ in (6.4) is actually necessary. This can be answered affirmatively, as the following example shows.

Example 6.11. Consider the Engel group $\mathbb{E}$ with underlying space $\mathbb{R}^4$ and its points denoted by $z = (x, y, u, v)$ along with the four pointwise linearly independent C^∞ smooth vector fields

$$X = \partial_x, \quad Y = \partial_y + x\cdot\partial_u + \tfrac{1}{2}x^2\cdot\partial_v, \quad U = \partial_u + x\cdot\partial_v \quad \text{and} \quad V = \partial_v.$$

Further consider the two non-involutive distributions $\mathcal{D}_2 := \operatorname{span}\{X, Y\}$ as well as $\mathcal{D}_3 := \operatorname{span}\{X, Y, U\}$ of rank 2 and 3 respectively. The reader may easily see for themselves that $\mathcal{D}_2 = \ker(\vartheta^1) \cap \ker(\vartheta^2)$, where $\vartheta^1 = x\cdot\mathrm{d}y - \mathrm{d}u$ and $\vartheta^2 = \frac{1}{2}x^2\cdot\mathrm{d}y - \mathrm{d}v$ and that $\mathcal{D}_3 = \ker(\vartheta)$, where $\vartheta = -\frac{1}{2}x^2\cdot\mathrm{d}y + x\cdot\mathrm{d}u - \mathrm{d}v$. It is also straightforward to see that $\mathrm{d}\vartheta^1 = \mathrm{d}x \wedge \mathrm{d}y$, $\mathrm{d}\vartheta^2 = x\cdot\mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}\vartheta = -x\cdot\mathrm{d}y + x\cdot\mathrm{d}u - \mathrm{d}v$.

In order to find an estimate for the size of $\tau(S, \mathcal{D}_2)$ for any C^2 smooth manifold $S \subseteq \mathbb{E}$, note that $\vartheta^1 \wedge \vartheta^2 \wedge \mathrm{d}\vartheta^1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v$ is a volume form. Therewith, Corollary 6.8 yields $\dim_{\mathrm{H}}(\tau(S, \mathcal{D}_2)) \le 1$.

To estimate the size of $\tau(S, \mathcal{D}_3)$ first note that any product of type $(\vartheta)^r \wedge (\mathrm{d}\vartheta)^s$, where $r > 1$ or $s > 1$, equals zero. On the other hand, letting $\zeta := \mathrm{d}y$, the product form $\vartheta \wedge \mathrm{d}\vartheta \wedge \zeta = -\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v$ is a volume form. Now again using Corollary 6.8, we get $\dim_{\mathrm{H}}(\tau(S, \mathcal{D}_3)) \le 2$.

7. Sharpness of Theorem 1.3

We point out that the examples from the previous section not only are useful applications of Theorem 1.3, but also show the sharpness of our estimates in the sense that there exist C^2 smooth manifolds S for which the lower and the upper estimates from Theorem 5.2 coincide. For Examples 6.4 to 6.10, this is an immediate consequence of Corollary 5.5. In the case of Example 6.11, consider the involutive subdistributions $\mathcal{D}_2' := \operatorname{span}\{X\} \subseteq \mathcal{D}_2$ and $\mathcal{D}_3' := \operatorname{span}\{Y, U\} \subseteq \mathcal{D}_3$ of rank 1 and 2 respectively.

It is therefore a natural question whether the lower and the upper estimates from Theorem 5.2 coincide for *all* C^1 smooth distributions. That this has to be answered negatively, however, follows already from Example 1.4: Whereas $\iota(\mathcal{D})$ is integer for any

C^1 smooth distribution by definition, the right side of (5.1) is not in the setting of the respective example. We are going to provide, in addition, an example for which (5.1) features *two strict* inequalities.

Another interesting issue related to the sharpness of Theorem 1.3 is the range of possible values for the right side of (1.2). In the setting of a one-codimensional distribution, a lower bound for the range is given by Theorem 5.4 and Corollary 5.5 respectively. As we shall see, any value between n and $2n$ can in fact be taken in the setting of a one-codimensional distribution on $\mathbb{R}^{2n+1}$, which is even realised by a suitably constructed manifold.

Example 7.1. Let $f:(0,1)\to\mathbb{R}$ be a C^∞ smooth function such that $f(x)=0$ if and only if x belongs to the middle third Cantor set $C\subseteq(0,1)$ and consider the corresponding (C^∞ smooth) distribution $\mathcal{D}:=\operatorname{span}\{\partial_x,\partial_y+f(x)\cdot\partial_t\}=\ker(\vartheta)$ on $U:=(0,1)^3$, where $\vartheta=f(x)\cdot\mathrm{d}y-\mathrm{d}t$ and where the points in U have coordinates $z=(x,y,t)$. Note that such an f exists by an application of Whitney decomposition and partition of unity; see [16, Proposition 2.3.4].

However, in order to avoid loss of control over $(f')^{-1}(0)$ in this general construction and to make our ideas more accessible to the reader, we shall construct such a function f explicitly: Starting from the 'bump function' $\psi:\mathbb{R}\to\mathbb{R}$,

$$\psi(x):=\begin{cases} e^{-1/(1-x^2)} & \text{if } -1<x<1 \\ 0 & \text{else}\end{cases},$$

we recursively define $\varphi_1(x):=\psi(6x-3)$ and $\varphi_{i+1}(x):=\varphi_i(3x)+\varphi_i(3x-2)$ for $i\in\mathbb{N}$. Then the series $f(x):=\sum_{i=1}^\infty 2^{-2^i}\varphi_i(x)$ converges and has the required properties, indeed, as it is not hard to be verified.

With respect to the upper estimate in (5.1), we shall mainly take into consideration A_2. In this regard, first note that $\mathrm{d}\vartheta_z=f'(x)\cdot\mathrm{d}x\wedge\mathrm{d}y$. Letting $X:=\operatorname{span}\{\partial_x,\partial_y\}$, we get $C':=C\times(0,1)^2\subseteq A_2$ and therewith $\max_{1\le k\le 3}\{\min\{\dim_{\mathrm{H}}(A_k\setminus A_{k+1}),k\}\}=2$, since $A_3=\emptyset$.

On the other hand, we shall see that $\iota(\mathcal{D})=1$: Assume by contradiction that $\iota(\mathcal{D})=2$. Then there exists a 2-dimensional C^2 smooth manifold $S\subseteq U$ that is everywhere tangent to $\mathcal{D}$. Without loss of generality, we may assume that S is the graph of a C^2 smooth function $g:U'\to(0,1)$, where $U'\subseteq(0,1)^2$ is an open set, i.e. $S=\{z=(x,y,t)\in U'\times(0,1):t=g(x,y)\}$. Since S is everywhere tangent to $\mathcal{D}$, we have $\ker(\vartheta_z)=T_z(S)=\operatorname{span}\{\partial_x+g_x(x,y)\cdot\partial_t,\partial_y+g_y(x,y)\cdot\partial_t\}$ for all $z=(x,y,t)\in S$. Hence $g_x(x,y)=0$ and $g_y(x,y)=f(x)$ for all $(x,y)\in U'$. Using the C^2 smoothness of S, we get $g_{xy}(x,y)=0=f'(x)=g_{yx}(x,y)$ for all $(x,y)\in U'$. To the contrary, the equality $f'(x)=0$ holds if and only if $x\in C\cup A$, where $A=\{\frac12,\frac16,\frac56,\frac1{18},\frac5{18},\frac{13}{18},\frac{17}{18},\dots\}$ is the set of the midpoints of the removed intervals in the construction of C. Now the fact that $C\cup A$ does not contain any open intervals contradicts the existence of S.

We conclude this example by indicating a (C^∞ smooth) manifold the Hausdorff dimension of whose tangency set lying strictly between the two estimates from (5.1): Let $S:=\{z\in(0,1)^3:t=\frac12\}$. Then $\tau(S,\mathcal{D})=S\cap C'$ and thus

$$\dim_{\mathrm{H}}(\tau(S,\mathcal{D}))=1+\dim_{\mathrm{H}}(C)=1+\frac{\log(2)}{\log(3)}.$$

We leave it to the reader to verify that actually for *any* given 2-dimensional C^2 smooth manifold S and the distribution $\mathcal{D}$ from the above example, the estimate $\dim_{\mathrm{H}}(\tau(S,\mathcal{D}))\le 1+\frac{\log(2)}{\log(3)}$ holds. This can be done e.g. by again interpreting S (locally) as the graph of a C^2 smooth function $g:U'\to(0,1)$, projecting its tangency set to the xy-plane and showing that $\dim_{\mathrm{H}}(\operatorname{proj}(\tau(S,\mathcal{D}))\setminus(C\times(0,1)))\le 1$, using Proposition 2.7.

The completeness and sharpness of our main theorem is also reflected in the following proposition, which generalises the idea of Example 1.4.

Proposition 7.2. *Let $n \in \mathbb{N}$ and let $0 \leq s \leq n$ be any real number. Then*

1) *there exist an open set $U \subseteq \mathbb{R}^{2n+1}$ and a C^∞ smooth distribution $\mathcal{D}$ of rank $2n$ on U such that the right side of (1.2) equals to $n+s$.*
2) *there exists a C^∞ smooth $2n$-dimensional manifold $S \subseteq U$ such that we have $\dim_{\mathrm{H}}(\tau(S, \mathcal{D})) = n + s$.*

Proof. The cases $s = 0$ as well as $s = n$ are covered by the contact, see Example 6.4, and the involutive cases respectively. The idea of this proof is to adapt the defining form of the contact distribution to our specific needs. Let us denote the points of $\mathbb{R}^{2n+1}$ by $z = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ and consider for $1 \leq s \leq n-1$, $s \in \mathbb{N}$, the smooth one-form $\vartheta := x_{s+1} \cdot \mathrm{d}y_{s+1} + \ldots + x_n \cdot \mathrm{d}y_n - \mathrm{d}t$ on $U := (0,1)^{2n} \times (-1,1) \subseteq \mathbb{R}^{2n+1}$. Observe that $\mathrm{d}\vartheta = \mathrm{d}x_{s+1} \wedge \mathrm{d}y_{s+1} + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$ and therefore

$$\vartheta \wedge (\mathrm{d}\vartheta)^{n-s} = \pm\, \mathrm{d}x_{s+1} \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_{s+1} \wedge \ldots \wedge \mathrm{d}y_n \wedge \mathrm{d}t$$

is a non-vanishing $(2n-2s+1)$-form on U. Now using (6.5), we immediately get $A_{n+s+1} \subseteq V_{1,n-s} = \emptyset$.

On the other hand, note that the distribution $\mathcal{D} := \ker(\vartheta)$ is

$$\mathcal{D} = \mathrm{span}\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_s}, \partial_{y_{s+1}} + x_{s+1} \cdot \partial_t, \ldots, \partial_{y_n} + x_n \cdot \partial_t\},$$

as immediately can be verified. Also note that $\mathcal{D}$ contains the involutive subdistribution $\mathcal{D}' = \mathrm{span}\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_s}\}$ of rank $n+s$, which directly implies $A_{n+s} = U$ and completes the proof of 1) in the particular case $s \in \mathbb{N}$.

The case of non-integer s is slightly more intricate, though can be handled as a combination of the integer case and Example 1.4 as follows: Define $l := \lfloor s \rfloor$ as well as $d := s - l$ and consider $U := (0,1)^n \times (0,1)^{l+1} \times (-1,1)^{n-l-1} \times (-1,1) \subseteq \mathbb{R}^{2n+1}$. We shall construct a set $A \subseteq U$ of Hausdorff dimension $n+s$ and find a one-form ϑ such that $A_{n+l+2} = \emptyset$ and $A_{n+l+1} = A$ for the corresponding distribution $\mathcal{D} = \ker(\vartheta)$.

Let $A := (0,1)^n \times (0,1)^l \times C \times \{0\}^{n-l-1} \times \{0\} \subseteq U$, where $C \subseteq (0,1)$ denotes a d-dimensional Cantor set, and note that $\dim_{\mathrm{H}}(A) = n + l + d = n + s$. Let further denote by $\varphi : \mathbb{R}^{2n+1} \to \mathbb{R}$ a Whitney function [16, Proposition 2.3.4] with respect to A with the properties: $\varphi \in C^\infty(\mathbb{R}^{2n+1})$, $\varphi(z) \geq 0$ for all $z \in \mathbb{R}^{2n+1}$ and $\varphi(z) = 0$ if and only if $z \in A$. Observe that by the choice of A, there exists a function φ with the above properties that does not depend on the first $n+l$ variables. Next define the function $f : \mathbb{R}^{2n+1} \to \mathbb{R}$, $f(z) := x_{l+1} \cdot \varphi(y_{l+1}, \ldots, y_n, t)$ and consider the associated C^∞ smooth one-form, pointwise defined by

$$\vartheta_z := f(z) \cdot \mathrm{d}y_{l+1} + x_{l+2} \cdot \mathrm{d}y_{l+2} + \ldots + x_n \cdot \mathrm{d}y_n - \mathrm{d}t.$$

In order to show $A_{n+l+2} = \emptyset$, observe that

$$\begin{aligned} \mathrm{d}\vartheta &= \frac{\partial f}{\partial y_{l+2}}\, \mathrm{d}y_{l+2} \wedge \mathrm{d}y_{l+1} + \ldots + \frac{\partial f}{\partial y_n}\, \mathrm{d}y_n \wedge \mathrm{d}y_{l+1} + \frac{\partial f}{\partial t}\, \mathrm{d}t \wedge \mathrm{d}y_{l+1} + \\ &\quad + \frac{\partial f}{\partial x_{l+1}}\, \mathrm{d}x_{l+1} \wedge \mathrm{d}y_{l+1} + \mathrm{d}x_{l+2} \wedge \mathrm{d}y_{l+2} + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n, \end{aligned}$$

which implies

$$\vartheta \wedge (\mathrm{d}\vartheta)^{n-l-1} = \omega \pm \omega_0,$$

where $\omega_0 = \mathrm{d}x_{l+2} \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_{l+2} \wedge \ldots \wedge \mathrm{d}y_n$ and ω is a $(2n-2l-1)$-form that is linearly independent of ω_0. Hence, using (6.5), we obtain $A_{n+l+2} \subseteq V_{1,n-l-1} = \emptyset$.

For the equality $A_{n+l+1} = A$, first consider $z \notin A$. Note that $\frac{\partial f}{\partial x_{l+1}}(z) = \varphi(z) \neq 0$ and that

$$\vartheta \wedge (\mathrm{d}\vartheta)^{n-l} = \omega' \pm \omega_0',$$

where $\omega_0' = \frac{\partial f}{\partial x_{l+1}}(z)\cdot \mathrm{d}x_{l+1} \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_{l+1} \wedge \ldots \wedge \mathrm{d}y_n$ and ω' is a $(2n-2l+1)$-form that is linearly independent of ω_0'. Since ω_0' does not vanish at z, we have $z \notin V_{1,n-l}$, and, again using (6.5), $z \notin A_{n+l+1}$. Thus $A_{n+l+1} \subseteq A$.

For the opposite inclusion $A \subseteq A_{n+l+1}$, consider $z \in A$. Since z is a global minimum of f, all partial derivatives of f vanish at z. This immediately implies that $\vartheta_z = x_{l+2}\cdot \mathrm{d}y_{l+2} + \ldots + x_n \cdot \mathrm{d}y_n - \mathrm{d}t$ and $\mathrm{d}\vartheta_z = \mathrm{d}x_{l+2} \wedge \mathrm{d}y_{l+2} + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$. Now for $X := \mathrm{span}\{\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_{l+1}}\} \in G(2n+1, n+l+1)$, we have $\mathrm{d}\vartheta_z|_X = 0$ and hence $z \in A_{n+l+1}$. ☐

Remark 7.3. Note that, additionally to the above assertions, $\iota(\mathcal{D}) = n + \lfloor s \rfloor$ holds for the distribution $\mathcal{D}$ in Proposition 7.2. This directly implies, as a consequence of Theorem 5.2, the existence of a C^2 smooth manifold $S \subseteq U$ with the property $n + \lfloor s \rfloor \leq \dim_{\mathrm{H}}(\tau(S, \mathcal{D}))$.

Remark 7.4. Also note that a similar statement to the one in the above proposition can be shown on an *even*-dimensional ambient space. In this setting, the roles of the two extremes are played by the symplectic and the involutive cases respectively.

8. Final remarks and questions

As we shall see in this final section, our results are not only sharp in the sense of Theorem 5.2. As already exemplified by the first author in [3], C^2 smoothness of the manifolds is crucial for our estimates. If merely $C^{1,\alpha}$ smoothness, $0 < \alpha < 1$, is assumed, the tangencies can grow significantly. The results in [3] are extended to higher codimensional cases in the sequel.

The second part of the section is devoted to open questions/problems related to our work. These include generalisations of our estimates into various directions, in particular to non-Euclidean metrics, e.g. in the context of Carnot groups. Additionally, we are going to present several approaches in order to find *lower estimates* for the size of tangencies of compact manifolds.

Towards the proposition on the necessity of C^2 smoothness, we consider a distribution $\mathcal{D}$ of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ that is translation invariant along $\mathbb{R}^m$ and start the discussion by explaining the connection between tangency sets and mappings with prescribed Jacobian matrix: First note that the translation invariance property for $\mathcal{D} = \mathrm{span}\{X_1, \ldots, X_n\}$ implies that the vector fields X_i, $1 \leq i \leq n$, are independent of $y \in \mathbb{R}^m$, i.e.

$$X_i(x, y) = X_i(x) = \partial_{x_i} + \sum_{j=1}^{m} c_{ij}(x)\cdot \partial_{y_j}.$$

Let us now consider an n-dimensional manifold $S \subseteq U$. Without loss of generality, we may assume that S is given as the graph Γ_f of a C^1 smooth mapping $f : Q \to \mathbb{R}^m$, where $Q \subseteq \mathbb{R}^n$ denotes the cube centred at the origin and with side length 1:

$$S = \{(x, f(x)) : x \in Q\}.$$

Equivalently, S may be seen as the level surface of the mapping $\varrho : \mathbb{R}^{n+m} \to \mathbb{R}^m$ given by $\varrho(x, y) = y - f(x)$: $S = \{(x, y) \in Q \times \mathbb{R}^m : \varrho(x, y) = 0\}$. A point $(x, y) \in Q \times \mathbb{R}^m$ is a tangency point of S if and only if $y = f(x)$ and $X_i(\varrho_j(x, y)) = 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and where $\varrho_j(x, y) = y_j - f_j(x)$ denotes the j-th component of $\varrho(x, y)$. This implies that $(x, f(x)) \in \tau(S, \mathcal{D})$ if and only if $c_{ij}(x) = \frac{\partial f_j}{\partial x_i}(x)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, or, equivalently,

$$(\mathrm{J}f)^T(x) = \begin{pmatrix} c_{11}(x) & \cdots & c_{1m}(x) \\ \vdots & \ddots & \vdots \\ c_{n1}(x) & \cdots & c_{nm}(x) \end{pmatrix}, \tag{8.1}$$

where $(\mathrm{J}f)$ denotes the Jacobian matrix of f. Using the notation $C(x)$ for the matrix from (8.1), we define $\tau'(f) := \{x \in Q : (\mathrm{J}f)^T(x) = C(x)\}$. Note that $\tau(f) = \tau'(f)$.

By these considerations, in order to obtain large tangencies, it is enough to prescribe $(\mathrm{J}f)(x)$ according to (8.1) on a large measure set within Q. The precise statement is formulated in the following

Lemma 8.1. *Let $Q \subseteq \mathbb{R}^n$ be the unit cube centred at the origin and $C : Q \to \mathbb{R}^{n\times m}$ a matrix of C^1 smooth functions $c_{ij} : Q \to \mathbb{R}$. Then*

1) *for any $\alpha > 0$ there exists a $C^{1,1}$ smooth mapping $f_\alpha : Q \to \mathbb{R}^m$ such that*
$$\dim_{\mathrm{H}}(\tau'(f_\alpha)) \geq n - \alpha.$$
2) *for any $\varepsilon > 0$ there exists $f_\varepsilon : Q \to \mathbb{R}^m$, $f_\varepsilon \in \bigcap_{0<\alpha<1} C^{1,\alpha}$ such that*
$$\mathcal{H}^n(\tau'(f_\varepsilon)) \geq 1 - \varepsilon.$$

Proof. In [3, Theorem 4.1], the first author proved the statement for the special case $m = 1$. But since f_α and f_ε respectively can be decomposed into a vector of m real-valued functions and the construction of the large measure sets in the proof of [3, Theorem 4.1] depends neither on C nor on f_α or f_ε respectively, the reader may assure themselves that the statement also holds for general m. □

We are now in position to formulate the main statement of this section.

Proposition 8.2. *Let $U \subseteq \mathbb{R}^{n+m}$ be an open set and let $\mathcal{D}$ be a C^1 smooth distribution of rank n on U that is translation invariant along $\mathbb{R}^m$. Then*

1) *for any $\alpha > 0$ there exists a $C^{1,1}$ smooth n-dimensional manifold $S_\alpha \subseteq U$ such that*
$$\dim_{\mathrm{H}}(\tau(S_\alpha, \mathcal{D})) \geq n - \alpha.$$
2) *there exists an n-dimensional manifold $S \subseteq U$ of smoothness $\bigcap_{0<\alpha<1} C^{1,\alpha}$ such that*
$$\mathcal{H}^n(S) < \infty \qquad \text{and} \qquad \mathcal{H}^n(\tau(S, \mathcal{D})) > 0.$$

Proof. Following the discussion in the beginning of this section, we may assume without loss of generality that S_α is the graph Γ_{f_α} of a $C^{1,1}$ smooth mapping $f_\alpha : Q \to \mathbb{R}^m$. Then $S_\alpha = \{(x, f_\alpha(x)) : x \in Q\}$ and, by the equality $\tau(f_\alpha) = \tau'(f_\alpha)$, we have that $\tau(S_\alpha, \mathcal{D}) = \{(x, f_\alpha(x)) : x \in \tau'(f_\alpha)\}$. Note that the projection $\Phi_\alpha^{-1} : Q \times \mathbb{R}^m \to Q$, where $\Phi_\alpha(x) = (x, f_\alpha(x))$, is Lipschitz. Thus [11] $\dim_{\mathrm{H}}(\tau(S_\alpha, \mathcal{D})) \geq \dim_{\mathrm{H}}(\tau'(f_\alpha))$. Now Lemma 8.1 implies 1). Applying the same arguments to S and f_ε respectively yields 2). □

The restriction in Proposition 8.2 to *translation invariant* distributions gives rise to our first

Problem 8.3. *State and prove a generalised version of Proposition 8.2 that holds for* any *C^1 smooth distribution.*

We are next turning our attention towards the generalisation of our main theorem into various directions. One of the probably most natural desired generalisations of our main results is formulated in

Problem 8.4. *Prove sharp estimates akin to Theorems 1.3 and 5.2 respectively for tangency sets of type $\tilde{\tau}(S, \mathcal{D})$, introduced in Definition 4.4.*

In the whole present article, we considered Hausdorff measures and dimensions with respect to the *Euclidean* metric on $\mathbb{R}^{n+m}$. In the context of Carnot groups, however, one might be more interested in corresponding results with respect to the sub-Riemannian *Carnot-Carathéodory* metric, to which we shall refer as the CC metric in the sequel.

Results in this direction have been presented both for Heisenberg groups [3] and later for general Carnot groups [17].

An upper estimate for $\dim_{\mathrm{CC}}(\tau(S,\mathcal{D}))$, where $\dim_{\mathrm{CC}}$ denotes the Hausdorff dimension with respect to the CC metric, can immediately be obtained using the dimension comparison formula [4] for general sets in Carnot groups: Recall [4] that in the setting of a Carnot group $\mathbb{G}$ with underlying space $\mathbb{R}^N$ and of homogeneous dimension Q, there exist (piecewise linear) functions $\beta_-, \beta_+ : [0,N] \to [0,Q]$ such that

$$\beta_-(\dim_{\mathrm{E}}(A)) \le \dim_{\mathrm{CC}}(A) \le \beta_+(\dim_{\mathrm{E}}(A)) \tag{8.2}$$

holds for any set $A \subseteq \mathbb{G}$, where $\dim_{\mathrm{E}}$ denotes the Hausdorff dimension with respect to the Euclidean metric. Combining (1.2) with (8.2) and denoting the right side of (1.2) by d for simplicity, we immediately get the estimate

$$\dim_{\mathrm{CC}}(\tau(S,\mathcal{D})) \le \beta_+(d) \le \dim_{\mathrm{CC}}(S).$$

Observe, however, that the above estimate might be not very strong for several reasons: First, we do not know much about its sharpness. And secondly, as a consequence of the conditions of Theorem 1.3, it holds only for C^2 smooth manifolds, whereas the original results [3, 17] apply for merely horizontally C^1 smooth manifolds. Recall at this point that a manifold S is called *horizontally* C^r, $r \in \mathbb{N}$, smooth if the respective property holds along the directions spanned by $\mathcal{D}$.

The desired generalisation of our main theorem is formulated in

Problem 8.5. *State and prove a generalisation of Theorem 1.3 for Carnot groups that estimates the size of the tangency sets of horizontally C^1 smooth manifolds in terms of their Hausdorff dimension with respect to the* CC *metric.*

Whereas our upper estimate according to Theorem 1.3 is general in the sense that it holds for any distribution and *any* manifold, the lower one according to Theorem 5.2 is not, since it only asserts the existence of *some* manifold with the respective property. It would be interesting to find a lower bound for the size of the tangency with respect to a given distribution that holds for *any* manifold.

To see that this is not precisely the interesting direction to be investigated, observe that if $\mathcal{D}$ is non-involutive, there exists a C^2 smooth manifold $S \subseteq \mathbb{R}^{n+m}$ such that $\tau(S,\mathcal{D}) \subsetneq S$. Since $\tau(S,\mathcal{D})$ is closed in S, the set $S \setminus \tau(S,\mathcal{D})$ is open in S and is a C^2 smooth manifold that does not contain any tangency points at all. Obviously, this results in any lower bound for the above problem being zero.

In order to avoid the present difficulties and obtain a more accurate statement, we shall consider a distribution on an open set $U \subseteq \mathbb{R}^{n+m}$ together with a *specific* n-dimensional manifold S. We shall then search a lower bound for the size of the tangencies with respect to all *embeddings* of S into U. The precise statement is formulated in

Problem 8.6. *Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and let S be an n-dimensional C^2 smooth manifold that can be embedded C^2 smoothly into U. Indicate a real number $0 \le l \le n$ such that*

$$\dim_{\mathrm{H}}(\tau(S',\mathcal{D})) \ge l \tag{8.3}$$

for all images S' of S under a C^2 smooth embedding into U, where $\tau(S',\mathcal{D})$ denotes the tangency set of S' with respect to $\mathcal{D}$.

In the case $l = 0$, find $n \in \mathbb{N}_0$, such that (8.3) *can be replaced by*

$$\mathrm{card}(\tau(S',\mathcal{D})) \ge n \tag{8.4}$$

or show that $n = \infty$.

Also indicate a C^2 smooth embedding of S into U such that the image S' yields equality in (8.3) *or* (8.4) *respectively.*

A natural generalisation of Problem 8.6 is formulated in

Problem 8.7. *Let $\mathcal{D}$ be a C^1 smooth distribution of rank n on an open set $U \subseteq \mathbb{R}^{n+m}$ and let S be an be an $(n+r)$-dimensional, $0 \leq r \leq m$, smooth manifold that can be embedded C^2 smoothly into U. Find estimates corresponding to* (8.3) *and* (8.4) *respectively for the size of the tangencies $\tilde{\tau}(S', \mathcal{D})$ and indicate a C^2 smooth embedding of S into U whose image realises equality.*

Although we are not able to solve Problems 8.6 and 8.7, we presume that the following two examples may indicate a track to their solution.

Example 8.8. Let ϑ be a one-form with constant coefficients on $\mathbb{R}^{n+1}$ and consider the (C^∞ smooth) distributions $\mathcal{D} := \ker(\vartheta)$ as well as $\mathcal{D}^\perp$. Let further $S \subseteq \mathbb{R}^{n+1}$ be a compact orientable C^2 smooth hypersurface without boundary. In order to estimate the size of the tangencies, observe that $\tau(S, \mathcal{D})$ and $\tilde{\tau}(S, \mathcal{D}^\perp)$ correspond to the critical sets of the projections $p^\perp : S \to \mathcal{D}^\perp(0)$ and $p : S \to \mathcal{D}(0)$ respectively.

Suitably identifying $\mathcal{D}^\perp(0)$ with the real axis, the projection $p^\perp$ becomes a height function and therewith a Morse function for almost all ϑ. Hence [21, p. 216], $p^\perp$ defines a homotopy spherical complex structure on S. Now according to [25], the number of critical points of $p^\perp$ is bounded from below by the Lusternik-Schnirelmann category of S.

For what concerns the tangency set $\tilde{\tau}(S, \mathcal{D}^\perp)$, first observe that p, featuring a compact domain and a non-compact codomain, is not surjective. Hence the degree [10] of p equals zero, which implies that the critical set of p separates S. Therefore $\dim_{\mathrm{H}}(\tilde{\tau}(S, \mathcal{D}^\perp)) \geq \dim(\tilde{\tau}(S, \mathcal{D}^\perp)) \geq n - 1$.

In the context of a *non-involutive* distribution, we have the following

Example 8.9. Consider the Heisenberg group $\mathbb{H}^n$ together with the horizontal distribution $\mathcal{H}$, as introduced in Example 6.5. Let further $S \subseteq \mathbb{H}^n$ be a compact orientable hypersurface without boundary of non-zero Euler characteristic $\chi(S)$. As we shall see, $\tau(S, \mathcal{H}) \neq \emptyset$: Consider the Gauß map [21] $n : S \to S^{2n}$ with respect to an arbitrary orientation of S on the one hand and the normalised cross product $N : \mathbb{H}^n \to S^{2n}$,

$$N(z) := \frac{X_1 \wedge \ldots \wedge X_n \wedge Y_1 \wedge \ldots \wedge Y_n}{|X_1 \wedge \ldots \wedge X_n \wedge Y_1 \wedge \ldots \wedge Y_n|}(z),$$

on the other hand. Then $\tau(S, \mathcal{H}) = \{z \in S : n(z) = N(z)\}$. Now while the degree of n amounts to $\frac{\chi(S)}{2} \neq 0$ according to [26], the normal unit vector field $N|_S : S \to S^{2n}$ is not onto and has degree zero therefore. Hence n and N are not homotopic, which results in $\tau(S, \mathcal{H}) \neq \emptyset$.

Remark 8.10. Observe that the assumption $\chi(S) \neq 0$ in Example 8.9 is essential. This follows e.g. from the fact [27] the tangency set of the standard torus in $\mathbb{H}^n$ (of zero Euler characteristic) being zero.

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