Decidability of Order-Based Modal Logics[☆]

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Abstract

Decidability of the validity problem is established for a family of many-valued modal logics, notably modal logics based on Gödel logics, where propositional connectives are evaluated locally at worlds according to the order of values in a complete chain and box and diamond modalities are evaluated as infima and suprema of values in (many-valued) Kripke frames. When the chain is infinite and the language is sufficiently expressive, the standard semantics for such a logic lacks the finite model property. It is shown here, however, that the finite model property does hold for a new equivalent semantics for the same logic. Decidability of the validity problem is also established for S5 versions of these logics that coincide with the one-variable fragments of first-order many-valued logics. In particular, a first proof is given of decidability and indeed of co-NP-completeness for the validity problem of the one-variable fragment of first-order Gödel logic.

[☆]A precursor to this paper, reporting preliminary results, has appeared as [7].

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1. Introduction

Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world and a crisp or many-valued accessibility relation between worlds to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness. Particular applications in the literature have included modelling fuzzy belief [14, 19], spatial reasoning with vague predicates [27], many-valued tense logics [11], fuzzy similarity measures [15], and the introduction of many-valued description logics, which, as in the classical case, may be understood as multi-modal logics (see, e.g., [4, 18, 29]).

Quite general approaches to many-valued modal logics, dealing mostly with decidability and axiomatization issues for finite-valued modal logics, are described in [5, 12, 13, 25]. For modal logics based on an infinite-valued semantics, two core families may be identified. Paradigmatic examples of the first family, many-valued modal logics of "magnitude" with propositional connectives interpreted by continuous functions over the real numbers, are based on the semantics of Łukasiewicz infinite-valued logic (see in particular [20]). The second family, and the topic of this paper, takes as paradigmatic examples, modal logics based on the semantics of infinite-valued Gödel logics (see [8, 9, 23]). These "order-based" modal logics are defined based on a complete linearly ordered set (chain) with operations depending only on the given order.

Let us remark that although Gödel logic is an intermediate logic, the many-valued modal logics considered here diverge quite considerably from the modal intermediate logics investigated in, e.g., [30], which use two accessibility relations for Kripke models, one for the modal operators and another for the intuitionistic connectives. We remark also that the modalities added to infinite-valued logics in [10, 17] represent truth stressers such as "very true" or "classically true" and, unlike the modalities considered here, may be interpreted as unary functions on the real unit interval.

The first main contribution of this paper is to establish the decidability of the validity problem for order-based modal logics (including Gödel modal logics) with certain natural sets of truth values: in particular, the real unit interval $[0, 1]$. The finite model property typically fails even for the box and diamond fragments of these logics. Decidability and PSPACE-completeness for the box and diamond fragments of Gödel modal logics is established in [23] using analytic Gentzen-style proof systems, but it is unclear if this methodology extends to the full logics. Here, an alternative Kripke semantics is provided for the logics that not only have the same valid formulas as the original semantics, but also admit the finite model property. The key idea of this new semantics is to restrict evaluations of modal formulas at a given world to a particular finite set of truth values.

The second main contribution of the paper is to establish decidability and co-NP-inclusion results for the validity problem of crisp order-based "S5" logics, i.e., order-based modal logics where accessibility is an equivalence relation. These logics may be viewed as one-variable fragments of first-order many-valued logics. In particular, we are able to answer positively the open problem of the decidability of the validity problem for the one-variable fragment of first-order Gödel logic (see, e.g., Problem (13) of Chapter 9 [16]) and show that it is co-NP-complete. This result matches the complexity of the one-variable fragment of classical first-order logic (equivalently, S5) and contrasts with the co-NEXPTIME-completeness of the one-variable fragment of first-order intuitionistic logic (equivalently, the intuitionistic modal logic MIPC) [22].

2. Order-Based Modal Logics

We consider "order-based" modal logics where propositional connectives are interpreted at individual worlds in an algebra consisting of a complete linearly ordered set of truth values and operations defined based only on the given order. Modalities $\Box$ and $\Diamond$ are defined using infima and suprema, respectively, according to either a (crisp) binary relation on the set of worlds or a binary mapping (many-valued relation) from worlds to values of the algebra. For convenience, we consider only finite algebraic languages, noting that to decide the validity of a formula we may in any case restrict to the language containing only operation symbols occurring in that formula.

2.1. Order-Based Algebras

Let $\mathcal{L}$ be a finite algebraic language that includes the binary operation symbols $\wedge$ and $\vee$ and constant symbols $\bot$ and $\top$ (to be interpreted by the usual lattice operations), and denote the set of constants of this language by $\mathrm{C}_{\mathcal{L}}$. An algebra $\mathbf{A}$ for $\mathcal{L}$ will be called *order-based* if it satisfies the following conditions:

(1) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}} \rangle$ is a *complete chain*: i.e., a bounded lattice with order $a \leq^{\mathbf{A}} b$ defined by $a \wedge^{\mathbf{A}} b = a$ that satisfies (i) $a \leq^{\mathbf{A}} b$ or $b \leq^{\mathbf{A}} a$ for all

$a, b \in A$, and (ii) $\bigwedge^{\mathbf{A}} B$ and $\bigvee^{\mathbf{A}} B$ exist in A for all $B \subseteq A$ (in particular, $\bot^{\mathbf{A}} = \bigvee^{\mathbf{A}} \emptyset$ and $\top^{\mathbf{A}} = \bigwedge^{\mathbf{A}} \emptyset$).

(2) For each operation symbol $\star$ of $\mathcal{L}$, the operation $\star^{\mathbf{A}}$ is definable in $\mathbf{A}$ by a quantifier-free formula in the first-order language with only $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$.

We also let $\mathrm{C}_{\mathcal{L}}^{\mathbf{A}}$ denote the finite set of constant operations $\{c^{\mathbf{A}} : c \in \mathrm{C}_{\mathcal{L}}\}$.

Note that, because $\mathbf{A}$ is a complete chain, an implication operation $\to^{\mathbf{A}}$ may always be introduced as the residual of $\wedge^{\mathbf{A}}$:

$$a \to^{\mathbf{A}} b \;=\; \bigvee\nolimits^{\mathbf{A}} \{c \in A : c \wedge^{\mathbf{A}} a \leq^{\mathbf{A}} b\} \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a \leq^{\mathbf{A}} b \\ b & \text{otherwise.} \end{cases}$$

Let $\varphi \leq \psi$ stand for $\varphi \wedge \psi \approx \varphi$ and $\varphi < \psi$ for $(\varphi \leq \psi)\ \&\ (\varphi \not\approx \psi)$, where $\&$ and $\Rightarrow$ are classical conjunction and implication, respectively. Then the implication operation $\to^{\mathbf{A}}$ is definable in $\mathbf{A}$ by the quantifier-free formula

$$F^{\to}(x, y, z) \;=\; ((x \leq y) \Rightarrow (z \approx \top))\ \&\ ((y < x) \Rightarrow (z \approx y)).$$

That is,

$$\mathbf{A} \models F^{\to}(a, b, c) \quad \Leftrightarrow \quad a \to^{\mathbf{A}} b = c.$$

In this paper, the implication connective $\to$ will always be interpreted by $\to^{\mathbf{A}}$. We will also make use of the defined negation connective $\neg\varphi = \varphi \to \bot$, interpreted by the unary operation

$$\neg^{\mathbf{A}} a \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a = \bot^{\mathbf{A}} \\ \bot^{\mathbf{A}} & \text{otherwise.} \end{cases}$$

Other operations covered by the order-based approach are

$$\Delta^{\mathbf{A}} a \;=\; \begin{cases} \top^{\mathbf{A}} & \text{if } a = \top^{\mathbf{A}} \\ \bot^{\mathbf{A}} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nabla^{\mathbf{A}} a \;=\; \begin{cases} \bot^{\mathbf{A}} & \text{if } a = \bot^{\mathbf{A}} \\ \top^{\mathbf{A}} & \text{otherwise} \end{cases}$$

where the first is the globalization or Baaz Delta operator, and the second is the Nabla operator (definable also as $\nabla^{\mathbf{A}} a = \neg^{\mathbf{A}}\neg^{\mathbf{A}} a$). Also definable by a suitable quantifier-free formulas is the dual-implication

$$a \leftarrow^{\mathbf{A}} b \;=\; \bigwedge\nolimits^{\mathbf{A}} \{c \in A : b \leq^{\mathbf{A}} a \vee^{\mathbf{A}} c\} \;=\; \begin{cases} \bot^{\mathbf{A}} & \text{if } b \leq^{\mathbf{A}} a \\ b & \text{otherwise.} \end{cases}$$

This operation may equivalently be defined as $a \leftarrow^{\mathbf{A}} b = \neg^{\mathbf{A}}\Delta^{\mathbf{A}}(b \to^{\mathbf{A}} a) \wedge^{\mathbf{A}} b$. In fact, it can be shown that if $A = [0, 1]$, then *any* operation definable in $\mathbf{A}$ by a quantifier-free formula in the first-order language with only $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$ is definable using the operations $\wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \to^{\mathbf{A}}, \Delta^{\mathbf{A}}$ and constant operations in $\mathrm{C}_{\mathcal{L}}^{\mathbf{A}}$. A similar result holds when A is any infinite set and $\mathrm{C}_{\mathcal{L}} = \{\bot, \top\}$.

2.2. Kripke Semantics

Let us fix an order-based algebra $\mathbf{A}$ for the language $\mathcal{L}$. We define order-based modal logics $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{K}(\mathbf{A})$ based on, respectively, standard (crisp) Kripke frames and Kripke frames with an accessibility relation taking values in A.

An $\mathbf{A}$-*frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ such that W is a non-empty set of *worlds* and $R\colon W \times W \to A$ is a binary $\mathbf{A}$-*accessibility relation* on W. If $Rxy \in \{\bot^{\mathbf{A}}, \top^{\mathbf{A}}\}$ for all $x, y \in W$, then R is called *crisp* and $\mathfrak{F}$ is called a *crisp* $\mathbf{A}$-*frame* or simply a *frame*. In this case, we often write $R \subseteq W \times W$ and Rxy to mean $Rxy = \top^{\mathbf{A}}$.

Now let $\mathcal{L}_{\Box\Diamond}$ be the language $\mathcal{L}$ with additional unary operation symbols (modal connectives) $\Box$ and $\Diamond$. The set of *formulas* $\mathrm{Fm}_{\Box\Diamond}^{\mathcal{L}}$ over $\mathcal{L}_{\Box\Diamond}$, denoted $\varphi, \psi, \ldots$ is defined inductively over a countably infinite set Var of propositional variables, denoted $p, q, \ldots$. *Subformulas* are defined as usual. We call formulas of the form $\Box\varphi$ and $\Diamond\varphi$ *box-formulas* and *diamond-formulas*, respectively, and fix the *length* of a formula φ, denoted by $\ell(\varphi)$, to be the number of symbols occurring in φ. We also let $\mathrm{Var}(\varphi)$ denote the set of variables occurring in the formula φ.

A $\mathsf{K}(\mathbf{A})$-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is an $\mathbf{A}$-frame and $V\colon \mathrm{Var} \times W \to A$ is a mapping, called a *valuation*. This valuation is extended to the mapping $V\colon \mathrm{Fm}_{\Box\Diamond}^{\mathcal{L}} \times W \to A$ by

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \;=\; \star^{\mathbf{A}}(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each n-ary operation symbol $\star$ of $\mathcal{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\nolimits^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\varphi, y) : y \in W\} \\ V(\Diamond\varphi, x) &= \bigvee\nolimits^{\mathbf{A}}\{Rxy \wedge^{\mathbf{A}} V(\varphi, y) : y \in W\}. \end{aligned}$$

A $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame. In this case, the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\nolimits^{\mathbf{A}}\{V(\varphi, y) : Rxy\} \\ V(\Diamond\varphi, x) &= \bigvee\nolimits^{\mathbf{A}}\{V(\varphi, y) : Rxy\}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ will be called *valid* in a $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) = \top^{\mathbf{A}}$ for all $x \in W$. If φ is valid in all L-models for some logic L, then φ is said to be L-*valid*, written $\models_{\mathsf{L}} \varphi$.

We now introduce some useful notation and terminology. A subset $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ will be called a *fragment* if it contains all constants of $\mathcal{L}$ and is closed with respect to taking subformulas. For a formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$, we let $\Sigma(\varphi)$ be the smallest fragment containing φ.

For a fragment Σ, let $\Sigma_\Box$ be the set of all box-formulas in Σ, and let $\Sigma_\Diamond$ be the set of all diamond-formulas in Σ. For a $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$, let us further define $V_x[\Delta] = \{V(\varphi, x) : \varphi \in \Delta\}$ for any $x \in W$ and $\Delta \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$, and

$$\Omega(\mathfrak{M}, \Sigma) = \bigcup_{x\in W} V_x[\Sigma_\Box \cup \Sigma_\Diamond].$$

Given a linearly ordered set $\langle P, \leq\rangle$ and $C \subseteq P$, a map $h\colon P \to P$ will be called a C-*order embedding* if it is an order-preserving embedding (i.e., $a \leq b$ if and only if $h(a) \leq h(b)$ for all $a, b \in P$) satisfying $h(c) = c$ for all $c \in C$. Moreover, h will be called B-*complete* for $B \subseteq P$ if whenever $\bigvee D \in B$ or $\bigwedge D \in B$ for some $D \subseteq P$, respectively,

$$h(\bigvee D) = \bigvee h[D] \quad \text{or} \quad h(\bigwedge D) = \bigwedge h[D].$$

The following lemma establishes the critical property of order-based modal logics that only the order of values taken by variables is important for determining the values of formulas and hence for checking validity.

Lemma 1. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model, $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ a fragment, and $h\colon A \to A$ a $\Omega(\mathfrak{M}, \Sigma)$-complete $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$-order embedding. Consider the $\mathsf{K}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V}\rangle$ with $\widehat{R}xy = h(Rxy)$ and $\widehat{V}(p, x) = h(V(p, x))$ for all $p \in \mathrm{Var}$ and $x, y \in W$. Then for all $\varphi \in \Sigma$ and $x \in W$:*

$$\widehat{V}(\varphi, x) = h(V(\varphi, x)).$$

Proof. We proceed by induction on the length of $\varphi \in \Sigma$. The case $\varphi \in \mathrm{Var} \cup \mathrm{C}_{\mathcal{L}}$ follows immediately from the definition of $\widehat{V}$. For the inductive step, suppose that $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for some operation symbol $\star$ of $\mathcal{L}$ and $\varphi_1, \ldots, \varphi_n \in \Sigma$. Then for some quantifier-free formula $F^\star(x_1, \ldots, x_n, y)$ in the language with $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$:

$$\mathbf{A} \models F^\star(a_1, \ldots, a_n, b) \quad \Leftrightarrow \quad \star^{\mathbf{A}}(a_1, \ldots, a_n) = b.$$

Moreover, as quantifier-free formulas are preserved by embeddings:

$$\mathbf{A} \models F^\star(a_1, \ldots, a_n, b) \quad \Leftrightarrow \quad \mathbf{A} \models F^\star(h(a_1), \ldots, h(a_n), h(b)).$$

So we may also conclude

$$\star^{\mathbf{A}}(h(a_1), \ldots, h(a_n)) = h(\star^{\mathbf{A}}(a_1, \ldots, a_n)).$$

Hence for all $x \in W$, using the induction hypothesis for the step from (1) to (2):

$$\begin{aligned}
\widehat{V}(\star(\varphi_1, \ldots, \varphi_n), x) &= \star^{\mathbf{A}}(\widehat{V}(\varphi_1, x), \ldots, \widehat{V}(\varphi_n, x)) && (1)\\
&= \star^{\mathbf{A}}(h(V(\varphi_1, x)), \ldots, h(V(\varphi_n, x))) && (2)\\
&= h(\star^{\mathbf{A}}(V(\varphi_1, x), \ldots, V(\varphi_n, x))) && (3)\\
&= h(V(\star(\varphi_1, \ldots, \varphi_n), x)). && (4)
\end{aligned}$$

If $\varphi = \Diamond\psi$ for some $\psi \in \Sigma$, then we obtain for all $x \in W$:

$$\begin{aligned}
\widehat{V}(\Diamond\psi, x) &= \bigvee^{\mathbf{A}}\{\widehat{R}xy \wedge^{\mathbf{A}} \widehat{V}(\psi, y) : y \in W\} && (1)\\
&= \bigvee^{\mathbf{A}}\{h(Rxy) \wedge^{\mathbf{A}} h(V(\psi, y)) : y \in W\} && (2)\\
&= \bigvee^{\mathbf{A}}\{h(Rxy \wedge^{\mathbf{A}} V(\psi, y)) : y \in W\} && (3)\\
&= h(\bigvee^{\mathbf{A}}\{Rxy \wedge^{\mathbf{A}} V(\psi, y) : y \in W\}) && (4)\\
&= h(V(\Diamond\psi, x)). && (5)
\end{aligned}$$

(1) to (2) follows from the definition of $\widehat{R}$ and the induction hypothesis, (2) to (3) follows because h is an order embedding, and (3) to (4) follows because h is $\Omega(\mathfrak{M}, \Sigma)$-complete and $\bigvee^{\mathbf{A}}\{Rxy \wedge^{\mathbf{A}} V(\psi, y) : y \in W\} = V(\Diamond\psi, x) \in V_x[\Sigma_\Diamond] \subseteq \Omega(\mathfrak{M}, \Sigma)$. The case $\varphi = \Box\psi$ is very similar. $\Box$

We now consider many-valued analogues of some useful notions and results from classical modal logic (see, e.g., [3]). For an $\mathbf{A}$-frame $\langle W, R\rangle$, let

$$R^+ = \{(x, y) \in W^2 : Rxy > \bot^{\mathbf{A}}\}, \quad R^+[x] = \{y \in W : R^+xy\} \text{ for } x \in W.$$

Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model. We call $\mathfrak{M}' = \langle W', R', V'\rangle$ a $\mathsf{K}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R' and V' are the restrictions to W' of R and V, respectively. In particular, given $X \subseteq W$, the $\mathsf{K}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* X is the smallest $\mathsf{K}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V'\rangle$ of $\mathfrak{M}$ satisfying $X \subseteq W'$ such that for all $x \in W'$, $y \in R^+[x]$ implies $y \in W'$.

Lemma 2. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a $\mathsf{K}(\mathbf{A})$-model and let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ be a generated $\mathsf{K}(\mathbf{A})$-submodel of $\mathfrak{M}$. Then $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $x \in \widehat{W}$ and $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$.*

Proof. We proceed by induction on $\ell(\varphi)$. The base case is trivial for any submodel $\mathfrak{M}'$ of $\mathfrak{M}$, so also for $\widehat{\mathfrak{M}}$. For the inductive step, the case where $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for some operation symbol $\star$ follows immediately using the induction hypothesis.

Suppose that $\varphi = \Box\psi$. Fix $x \in \widehat{W}$ and note that for any $y \in W \setminus \widehat{W}$, we have $Rxy = \bot^{\mathbf{A}}$; observe also that $\bot^{\mathbf{A}} \to^{\mathbf{A}} a = \top^{\mathbf{A}}$ for all $a \in A$. Hence, excluding all worlds $y \in W$ such that $Rxy = \bot^{\mathbf{A}}$ does not change the value of $\bigwedge^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\}$. So, using the induction hypothesis,

$$\begin{aligned} V(\Box\psi, x) &= \bigwedge\nolimits^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in \widehat{W}\} && (6) \\ &= \bigwedge\nolimits^{\mathbf{A}}\{\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) : y \in \widehat{W}\} && (7) \\ &= \widehat{V}(\Box\psi, x). && (8) \end{aligned}$$

The case where $\varphi = \Diamond\psi$ is very similar. □

A $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V \rangle$ is called a $\mathsf{K}(\mathbf{A})$-*tree-model* if $\langle W, R^+ \rangle$ is a tree. In this case, the *height* $\mathrm{hg}(\mathfrak{M})$ of $\mathfrak{M}$ is the height of the tree (possibly ∞) and the height $\mathrm{hg}(x)$ of $x \in W$ is the height of x in the tree.

Lemma 3. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a $\mathsf{K}(\mathbf{A})$-model, $x_0 \in W$, and $k \in \mathbb{Z}^+$. Then there exists a $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}_0$ and $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ with $\ell(\varphi) \leq k$. Moreover, if $\mathfrak{M}$ is a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model, then so is $\widehat{\mathfrak{M}}$.*

Proof. Consider the $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M}' = \langle W', R', V' \rangle$ obtained by unravelling at the world x_0; i.e., let

$$\begin{aligned} W' &= \{(x_0, \ldots, x_n) \in W^{n+1} : R^+ x_i x_{i+1} \text{ for } 0 \leq i < n\} \\ R'yz &= \begin{cases} Rx_n x_{n+1} & \text{if } y = (x_0, \ldots, x_n),\ z = (x_0, \ldots, x_{n+1}) \\ \bot^{\mathbf{A}} & \text{otherwise.} \end{cases} \\ V'(p, (x_0, \ldots, x_n)) &= V(p, x_n). \end{aligned}$$

Clearly, $\mathfrak{M}'$ is a $\mathsf{K}(\mathbf{A})$-tree-model with root $\widehat{x}_0 = (x_0)$. Now let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ be the $\mathsf{K}(\mathbf{A})$-tree-submodel of $\mathfrak{M}'$ defined by cutting $\mathfrak{M}'$ at height k; i.e., let $\widehat{W} =$

$\{x \in W' : \mathrm{hg}(x) \leq k\}$ and let $\widehat{R}$ and $\widehat{V}$ be the restrictions of R' and V' to $\widehat{W} \times \widehat{W}$ and $\mathrm{Var} \times \widehat{W}$, respectively. A straightforward induction on $\ell(\varphi)$ shows that for all $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ and $n \in \mathbb{N}$ such that $\ell(\varphi) \leq k - n$, $\widehat{V}(\varphi, (x_0, \ldots, x_n)) = V(\varphi, x_n)$. In particular, $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ with $\ell(\varphi) \leq k$. $\Box$

2.3. Gödel Modal Logics

Consider the standard infinite-valued Gödel algebra

$$\mathbf{G} = \langle [0,1], \wedge, \vee, \to, \bot, \top \rangle.$$

The logics $\mathsf{K}(\mathbf{G})$ and $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$ are the "Gödel modal logics" GK and GK^{C} studied in [5, 8, 9, 23]. Axiomatizations of the box and diamond fragments of GK are obtained in [8] by extending an axiomatization of Gödel logic (intuitionistic logic plus the prelinearity axiom schema $(\varphi \to \psi) \vee (\psi \to \varphi)$) with, respectively,

$$\begin{array}{lcl} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & & \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi) \\ \neg\neg\Box\varphi \to \Box\neg\neg\varphi & \text{and} & \Diamond\neg\neg\varphi \to \neg\neg\Diamond\psi \\ \varphi \;/\; \Box\varphi & & \neg\Diamond\bot \\ & & \varphi \to \psi \;/\; \Diamond\varphi \to \Diamond\psi. \end{array}$$

An axiomatization of the full logic GK is obtained in [9] by extending the union of these axiomatizations with the Fischer Servi axioms (see [28])

$$\begin{array}{l} \Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi) \\ (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi). \end{array}$$

It is also shown in [9] that GK coincides with the extension of the intuitionistic modal logic IK with the prelinearity axiom schema.

The box fragment of GK^{C} coincides with the box fragment of GK [8], and the diamond fragment of GK^{C} is axiomatized in [23] as an extension of the diamond fragment of GK with

$$\varphi \vee (\psi_1 \to \psi_2) \;/\; \Diamond\varphi \vee (\Diamond\psi_1 \to \Diamond\psi_2).$$

However, no axiomatization has yet been found for the full logic GK^{C}.

A more general perspective on Gödel modal logics is obtained by considering the family of Gödel logics defined by algebras $\mathbf{A} = \langle A, \wedge, \vee, \to, \bot, \top \rangle$ satisfying $\{0,1\} \subseteq A \subseteq [0,1]$. It is not hard to show that for finite A, the sets of valid formulas of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ depend only on the cardinality of A and are decidable. (In fact, validity in any of the minimum many-valued modal logics based on a finite residuated lattice considered in [5] is decidable.) Any infinite A gives the same set of valid propositional formulas; however, as shown by Baaz et al. in [1], there are countably infinitely many different first-order Gödel logics. We conjecture that this result holds also for Gödel modal logics, but restrict our attention in this paper to the real unit interval $[0, 1]$ and two further "natural" choices,

$$G_{\downarrow} = \{0\} \cup \{\tfrac{1}{n} \mid n \in \mathbb{Z}^+\} \quad \text{and} \quad G_{\uparrow} = \{1 - \tfrac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{1\},$$

where $\mathbf{G}_{\downarrow} = \langle G_{\downarrow}, \wedge, \vee, \to, \bot, \top \rangle$ and $\mathbf{G}_{\uparrow} = \langle G_{\downarrow}, \wedge, \vee, \to, \bot, \top \rangle$. Clearly, order-based algebras with these universes are isomorphic to algebras with universes $\{-n : n \in \mathbb{N}\} \cup \{-\infty\}$ and $\mathbb{N} \cup \{\infty\}$.

The logics $\mathsf{K}(\mathbf{G})$, $\mathsf{K}(\mathbf{G}_{\uparrow})$, $\mathsf{K}(\mathbf{G}_{\downarrow})$ and their crisp counterparts are all distinct. The formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in the logics based on $\mathbf{G}_{\uparrow}$, but not in those based on $\mathbf{G}$ or $\mathbf{G}_{\downarrow}$, while $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$ is valid in the logics based on $\mathbf{G}_{\downarrow}$ but not those based on $\mathbf{G}$. Moreover, the formula $\neg\neg\Diamond p \to \Diamond\neg\neg p$ is valid in any of the crisp logics, but not in the non-crisp versions.

2.4. The Finite Model Property

Let us call an L-model for a logic L *countable* or *finite* if its set of worlds is countable or finite, respectively. We say that a logic L has the *finite model property* if validity in the logic coincides with validity in all finite L-models. It is shown in [8] that the diamond fragment of GK has the finite model property and, using this result, that validity in the fragment is decidable. Moreover, in one of our core cases, we are able to show that the full logics have the finite model property:

Theorem 4. *$\mathsf{K}(\mathbf{G}_{\uparrow})$ and $\mathsf{K}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ have the finite model property.*

Proof. By Lemma 3, it suffices to show that if $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is not valid in some $\mathsf{K}(\mathbf{G}_{\uparrow})$-tree-model $\mathfrak{M}$ of finite height, then φ is not valid in some finite $\mathsf{K}(\mathbf{G}_{\uparrow})$-tree-model $\widehat{\mathfrak{M}}$ of finite height, where $\widehat{\mathfrak{M}}$ is crisp if $\mathfrak{M}$ is crisp.

Suppose that $V(\varphi, x) < 1$ for some $\mathsf{K}(\mathbf{G}_{\uparrow})$-tree-model of finite height $\mathfrak{M} = \langle W, R, V \rangle$ with root x. We define a $\mathsf{K}(\mathbf{G}_{\uparrow})$-model $\mathfrak{M}' = \langle W, R', V' \rangle$ (that is crisp if $\mathfrak{M}$ is crisp) using $h\colon \mathbf{G}_{\uparrow} \to \mathbf{G}_{\uparrow}$ defined by

$$h(a) = \begin{cases} a & \text{if } a \leq V(\varphi, x) \\ 1 & \text{otherwise.} \end{cases}$$

Let $R'yz = h(Ryz)$ for all $y, z \in W$ and $V'(p, y) = h(V(p, y))$ for all $y \in W$ and $p \in \mathrm{Var}$. We prove that $V'(\psi, y) = h(V(\psi, y))$ for all $y \in W$ and $\psi \in \Sigma(\varphi)$

by induction on $\ell(\psi)$. The base case follows by definition (recalling that the only constants are $\bot$ and $\top$). For the induction step, the modal cases follow as in Lemma 1 and the propositional cases follow by observing that h is a Heyting algebra homomorphism.

Finally, we obtain a finite tree-submodel $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ of $\mathfrak{M}'$ with root x such that $\widehat{V}(\psi, x) < 1$ for all $\psi \in \Sigma(\varphi)$, proceeding by induction on $\mathrm{hg}(\mathfrak{M}')$. Omitting the details, the key observation is that the value of each box or diamond formula in $\Sigma(\varphi)$ at height n is witnessed by a formula at height $n+1$ (even if this value is 1) and hence only finite branching is required in the submodel. $\square$

The finite model property does not hold for Gödel modal logics with universe $[0,1]$ or $G_\downarrow$, however, or even $G_\uparrow$ if we add also the connective Δ to the language.

Theorem 5. *Suppose that either (i) the universe of $\mathbf{A}$ is $[0,1]$ or $G_\downarrow$ and the language contains $\rightarrow$, or (ii) the universe of $\mathbf{A}$ is $G_\uparrow$ and the language contains $\rightarrow$ and Δ. Then neither $\mathsf{K}(\mathbf{A})$ nor $\mathsf{K}(\mathbf{A})^\mathsf{C}$ has the finite model property.*

Proof. For (i), we follow [8] where it is shown that the following formula provides a counterexample to the finite model property of GK and GK^C:

$$\varphi = \Box\neg\neg p \rightarrow \neg\neg\Box p.$$

Just observe that φ is valid in all finite $\mathsf{K}(\mathbf{A})$-models, but not in the infinite $\mathsf{K}(\mathbf{A})^\mathsf{C}$-model $\langle \mathbb{Z}^+, R, V \rangle$ where $Rxy = 1$ for all $x, y \in \mathbb{Z}^+$ and $V(p, x) = \frac{1}{x}$ for all $x \in \mathbb{Z}^+$. Hence neither $\mathsf{K}(\mathbf{A})$ nor $\mathsf{K}(\mathbf{A})^\mathsf{C}$ has the finite model property.

Similarly, for (ii), the formula

$$\psi = \Delta\Diamond p \rightarrow \Diamond\Delta p$$

is valid in all finite $\mathsf{K}(\mathbf{A})$-models, but not in the infinite $\mathsf{K}(\mathbf{A})^\mathsf{C}$-model $\langle \mathbb{Z}^+, R, V \rangle$ where $Rxy = 1$ for all $x, y \in \mathbb{Z}^+$ and $V(p, x) = \frac{x-1}{x}$ for all $x \in \mathbb{Z}^+$. $\square$

Let us remark also that decidability and indeed PSPACE-completeness of validity in the box and diamond fragments of both GK and GK^C has been established in [23] using analytic Gentzen-style proof systems, but that decidability of validity in the full logics GK and GK^C has remained open.

3. A New Semantics for the Modal Operators

Let us assume again that $\mathbf{A}$ is an order-based algebra and that A is $[0,1]$, $G_{\downarrow}$, or $G_{\uparrow}$. Consider again the failure of the finite model property for GK^{C} established in the proof of Theorem 5. For a GK^{C}-model to render $\varphi = \Box\neg\neg p \to \neg\neg\Box p$ invalid at a world x, there must be values of p at worlds accessible to x that form an infinite descending sequence tending to but never reaching 0. This ensures that the infinite model falsifies φ, but also that no particular world acts as a "witness" to the value of $\Box p$. Here, we redefine models to allow only a finite number of values at each world that can be taken by box-formulas and diamond-formulas. A formula such as $\Box p$ can then be "witnessed" at a world where the value of p is merely "sufficiently close" to the value of $\Box p$.

We define a $\mathsf{FK}(\mathbf{A})$-*model* as a five-tuple $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$ such that $\langle W, R, V\rangle$ is a $\mathsf{K}(\mathbf{A})$-model and $T_{\Box}\colon W \to \mathcal{P}(A)$ and $T_{\Diamond}\colon W \to \mathcal{P}(A)$ are functions satisfying for each $x \in W$:

(i) $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}} \subseteq T_{\Box}(x) \cap T_{\Diamond}(x)$.

(ii) If A is $[0,1]$, then $T_{\Box}(x) = T_{\Diamond}(x)$ is finite.

(iii) If A is $G_{\downarrow}$, then for some $m \in \mathbb{Z}^{+}$,

$$T_{\Box}(x) = \{0, \tfrac{1}{m}, \tfrac{1}{m-1}, \ldots, \tfrac{1}{2}, 1\} \quad \text{and} \quad T_{\Diamond}(x) = G_{\downarrow}.$$

(iv) If A is $G_{\uparrow}$, then for some $m \in \mathbb{Z}^{+}$,

$$T_{\Box}(x) = G_{\uparrow} \quad \text{and} \quad T_{\Diamond}(x) = \{0, \tfrac{1}{2}, \ldots, \tfrac{m-1}{m}, 1\}.$$

The valuation V is extended to the mapping $V\colon \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond} \times W$ by

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \;=\; \star^{\mathbf{A}}(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each n-ary operational symbol $\star$ of $\mathcal{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\nolimits^{\mathbf{A}}\{r \in T_{\Box}(x) : r \leq^{\mathbf{A}} \bigwedge\nolimits^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\nolimits^{\mathbf{A}}\{r \in T_{\Diamond}(x) : r \geq^{\mathbf{A}} \bigvee\nolimits^{\mathbf{A}}\{Rxy \wedge^{\mathbf{A}} V(\varphi, y) : y \in W\}\}. \end{aligned}$$

As before, an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame, and the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned}V(\Box\varphi, x) &= \bigvee^{\mathbf{A}}\{r \in T_\Box(x) : r \leq^{\mathbf{A}} \bigwedge^{\mathbf{A}}\{V(\varphi, y) : Rxy\}\}\\ V(\Diamond\varphi, x) &= \bigwedge^{\mathbf{A}}\{r \in T_\Diamond(x) : r \geq^{\mathbf{A}} \bigvee^{\mathbf{A}}\{V(\varphi, y) : Rxy\}\}.\end{aligned}$$

A formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is *valid* in $\mathfrak{M}$ if $V(\varphi, x) = \top^{\mathbf{A}}$ for all $x \in W$.

Note that this revision does not affect the value of box-formulas when A is $G_\uparrow$ or diamond-formulas when A is $G_\downarrow$. In other cases, the interpretations of box-formulas and diamond-formulas depend on the choice of $T_\Box$ and $T_\Diamond$. However, always $V(\Box\varphi, x) \in T_\Box(x)$ and $V(\Diamond\varphi, x) \in T_\Diamond(x)$.

Let us extend some of our previous notions and properties to $\mathsf{FK}(\mathbf{A})$-models. Given an $\mathsf{FK}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$, we call $\mathfrak{M}' = \langle W', R', V', T'_\Box, T'_\Diamond\rangle$ an $\mathsf{FK}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R', V', $T'_\Box$, and $T'_\Diamond$ are the restrictions to W' of R, V, $T_\Box$, and $T_\Diamond$, respectively. As before, given $X \subseteq W$, the $\mathsf{FK}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* X is the smallest $\mathsf{FK}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V', T'_\Box, T'_\Diamond\rangle$ of $\mathfrak{M}$ satisfying $X \subseteq W'$ such that for all $x \in W'$, $y \in R^+[x]$ implies $y \in W'$. Lemmas 2 and 3 then extend to $\mathsf{FK}(\mathbf{A})$-models as follows with minimal changes in the proofs.

Lemma 6. *Let $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ be an $\mathsf{FK}(\mathbf{A})$-model.*

(a) *Let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ be a generated $\mathsf{FK}(\mathbf{A})$-submodel of $\mathfrak{M}$. Then $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $x \in \widehat{W}$, and $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$.*

(b) *Given any $x \in W$ and $k \in \mathbb{Z}^+$, there exists an $\mathsf{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ with root $\widehat{x}$ and $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ with $\ell(\varphi) \leq k$. Moreover, if $\mathfrak{M}$ is an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-model, then so is $\widehat{\mathfrak{M}}$.*

Observe that there are very simple finite $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-counter-models when $A = [0, 1]$ for the formula $\Box\neg\neg p \to \neg\neg\Box p$. Consider $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ with $W =$

$\{a\}$, $Raa = 1$, $T_\Box(a) = T_\Diamond(a) = \mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$, and $0 < V(p,a) < \min(\mathrm{C}^{\mathbf{A}}_{\mathcal{L}} \setminus \{0\})$. Then

$$\begin{aligned}
V(\Box\neg\neg p, a) &= \bigvee^{\mathbf{A}}\{r \in \mathrm{C}^{\mathbf{A}}_{\mathcal{L}} : r \leq^{\mathbf{A}} \bigwedge^{\mathbf{A}}\{V(\neg\neg p, y) : Ray\}\} \\
&= \bigvee^{\mathbf{A}}\{r \in \mathrm{C}^{\mathbf{A}}_{\mathcal{L}} : r \leq^{\mathbf{A}} V(\neg\neg p, a)\} \\
&= \bigvee^{\mathbf{A}}\{r \in \mathrm{C}^{\mathbf{A}}_{\mathcal{L}} : r \leq^{\mathbf{A}} 1\} \\
&= 1 \\
V(\neg\neg\Box p, a) &= \neg^{\mathbf{A}}\neg^{\mathbf{A}}\bigvee^{\mathbf{A}}\{r \in \mathrm{C}^{\mathbf{A}}_{\mathcal{L}} : r \leq^{\mathbf{A}} \bigwedge^{\mathbf{A}}\{V(p, y) : Ray\}\} \\
&= \neg^{\mathbf{A}}\neg^{\mathbf{A}}\bigvee^{\mathbf{A}}\{r \in \mathrm{C}^{\mathbf{A}}_{\mathcal{L}} : r \leq^{\mathbf{A}} V(p, a)\} \\
&= \neg^{\mathbf{A}}\neg^{\mathbf{A}}0 \\
&= 0 \\
V(\Box\neg\neg p \to \neg\neg\Box p, a) &= V(\Box\neg\neg p, a) \to^{\mathbf{A}} V(\neg\neg\Box p, a) \\
&= 1 \to^{\mathbf{A}} 0 \\
&= 0.
\end{aligned}$$

The same formula fails in a similar finite $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-model when $A = G_\downarrow$, and $\Delta\Diamond p \to \Diamond\Delta p$ fails in a similar $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-model when $A = G_\uparrow$.

In fact, we may "prune" (i.e., remove branches from) any $\mathsf{FK}(\mathbf{A})$-tree-model of finite height where $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is not valid in such a way that φ is not valid in the new finite $\mathsf{FK}(\mathbf{A})$-model. It then follows by part (b) of Lemma 6 that $\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ have the finite model property.

Lemma 7. *Let $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ be a finite fragment. Then for any $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ of finite height with root x, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T_\Box}, \widehat{T_\Diamond}\rangle$ with $\langle \widehat{W}, \widehat{R}, \widehat{V}\rangle \subseteq \langle W, R, V\rangle$, root $x \in \widehat{W}$, and $|\widehat{W}| \leq |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$.*

Proof. We prove the lemma by induction on $\mathrm{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and it suffices to define $\widehat{\mathfrak{M}} = \mathfrak{M}$.

For the induction step $\mathrm{hg}(\mathfrak{M}) = n + 1$, consider for each $y \in R^+[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y}\rangle$ of $\mathfrak{M}$ generated by $\{y\}$. Each $\mathfrak{M}_y$ is a $\mathsf{FK}(\mathbf{A})$-tree-model of finite height with root y and $\mathrm{hg}(\mathfrak{M}_y) \leq n$. Hence, by the induction hypothesis, for each $y \in R^+[x]$, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y, \widehat{T}_{\Box y}, \widehat{T}_{\Diamond y}\rangle$ with $\langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y\rangle \subseteq \langle W_y, R_y, V_y\rangle$ and root $y \in \widehat{W}_y$ such that $|\widehat{W}_y| \leq |\Sigma|^n$ and $\widehat{V}_y(\varphi, y) = V_y(\varphi, y)$ ($= V(\varphi, y)$, by Lemma 6(a)) for all $\varphi \in \Sigma$.

We now choose a finite number of appropriate $y \in R^+[x]$ in order to build our finite $\mathsf{FK}(\mathbf{A})$-model $\widehat{\mathfrak{M}}$. To this end, we define $\{\alpha_0, \ldots, \alpha_m\} \subseteq [0, 1]$ such that

(i) $0 = \alpha_0 < \ldots < \alpha_m = 1$.

(ii) If A is $[0,1]$, then $\{\alpha_0, \ldots, \alpha_m\} = V_x[\Sigma_\Box \cup \Sigma_\Diamond \cup \mathrm{C}_\mathcal{L}]$.

(iii) If A is $G_\downarrow$, then $\{\alpha_0, \ldots, \alpha_m\} = \{0, \min(V_x[\Sigma_\Box \cup \mathrm{C}_\mathcal{L}] \setminus \{0\}), \ldots, \frac{1}{2}, 1\}$.

(iv) If A is $G_\uparrow$, then $\{\alpha_0, \ldots, \alpha_m\} = \{0, \frac{1}{2}, \ldots, \max(V_x[\Sigma_\Diamond \cup \mathrm{C}_\mathcal{L}] \setminus \{1\}), 1\}$.

Consider $\Box\psi \in \Sigma_\Box$. If $V(\Box\psi, x) = \alpha_i$ for some $i \in \{0, \ldots, m\}$, then we choose a world $y_{\Box\psi} \in R^+[x]$ such that for $\alpha_i < 1$, also $Rxy_{\Box\psi} \to^{\mathbf{A}} V(\psi, y_{\Box\psi}) < \alpha_{i+1}$. If $V(\Box\psi, x) \notin \{\alpha_0, \ldots, \alpha_m\}$, which is only possible if $A = G_\uparrow$, then we choose a world $y_{\Box\psi} \in R^+[x]$ such that $Rxy_{\Box\psi} \to^{\mathbf{A}} V(\psi, y_{\Box\psi}) = V(\Box\psi, x)$.

Similarly, for each $\Diamond\psi \in \Sigma_\Diamond$ such that $V(\Diamond\psi, x) = \alpha_i$ for some $i \in \{0, \ldots, m\}$, we choose a world $y_{\Diamond\psi} \in R^+[x]$ such that for $\alpha_i > 0$, also $Rxy_{\Diamond\psi} \wedge^{\mathbf{A}} V(\psi, y_{\Diamond\psi}) > \alpha_{i-1}$. If $V(\Box\psi, x) \notin \{\alpha_0, \ldots, \alpha_m\}$, which in this case is only possible if $A = G_\downarrow$, we choose $y_{\Diamond\psi} \in R^+[x]$ such that $Rxy_{\Diamond\psi} \wedge^{\mathbf{A}} V(\psi, y_{\Diamond\psi}) = V(\Diamond\psi, x)$.

Now let $Y = \{y_\varphi \in R^+[x] : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$, noting that $|Y| \leq |\Sigma_\Box \cup \Sigma_\Diamond| < |\Sigma|$. We define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ where

$$\widehat{W} = \{x\} \cup \bigcup_{y \in Y} \widehat{W}_y,$$

$\widehat{R}$ and $\widehat{V}$ are R and V restricted to $\widehat{W} \times \widehat{W}$, respectively, and $\widehat{T}_\Box$ and $\widehat{T}_\Diamond$ satisfy

(i) $\widehat{T}_\Box(z) = \widehat{T}_{\Box y}(z)$ and $\widehat{T}_\Diamond(z) = \widehat{T}_{\Diamond y}(z)$ for all $y \in Y$ and $z \in \widehat{W}_y$.

(ii) If A is $[0,1]$, then $\widehat{T}_\Box(x) = \widehat{T}_\Diamond(x) = \{\alpha_0, \ldots, \alpha_m\}$.

(iii) If A is $G_\downarrow$, then $\widehat{T}_\Diamond(x) = G_\downarrow$ and $\widehat{T}_\Box(x) = \{\alpha_0, \ldots, \alpha_m\}$.

(iv) If A is $G_\uparrow$, then $\widehat{T}_\Box(x) = G_\uparrow$ and $\widehat{T}_\Diamond(x) = \{\alpha_0, \ldots, \alpha_m\}$.

Note that $\langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq \langle W, R, V \rangle$ and $|\widehat{W}| \leq |Y||\Sigma|^n + 1 < |\Sigma||\Sigma|^n = |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$. We show now that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$, proceeding by induction on $\ell(\varphi)$.

The base case follows directly from the definition of $\widehat{V}$. For the inductive step, the non-modal cases follow directly using the induction hypothesis. Suppose now that $\varphi = \Box\psi$, noting that the case $\varphi = \Diamond\psi$ is very similar. There are three cases. Suppose first that $Rxy \to^{\mathbf{A}} V(\psi, y) = 1$ for all $y \in R^+[x]$. This implies that

$Rxy \leq V(\psi, y)$ for all $y \in \widehat{R}^{+}[x] \subseteq R^{+}[x]$ and thus $\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) = 1$ for all $y \in \widehat{W}$. But $1 \in \widehat{T}_{\Box}(x)$, so

$$\widehat{V}(\Box\psi, x) = \bigvee^{\mathbf{A}}\{r \in \widehat{T}_{\Box}(x) : r \leq \bigwedge^{\mathbf{A}}\{\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = 1.$$

For the second case, suppose that for some $i \in \{0, \dots, m-1\}$:

$$V(\Box\psi, x) = \bigvee^{\mathbf{A}}\{r \in T_{\Box}(x) : r \leq \bigwedge^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\}\} = \alpha_i.$$

It follows that

$$\bigwedge^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\} \in [\alpha_i, \alpha_{i+1}).$$

But $\widehat{W} \subseteq W$ and $\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) = Rxy \to^{\mathbf{A}} V(\psi, y)$ for each $y \in \widehat{W}$, so also

$$\bigwedge^{\mathbf{A}}\{\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) : y \in \widehat{W}\} \geq \bigwedge^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\} \geq \alpha_i.$$

Because of our choice of $y_{\Box\psi} \in \widehat{W}$, we have

$$\widehat{R}xy_{\Box\psi} \to^{\mathbf{A}} \widehat{V}(\psi, y_{\Box\psi}) = Rxy_{\Box\psi} \to^{\mathbf{A}} V(\psi, y_{\Box\psi}) < \alpha_{i+1}.$$

Hence $\alpha_i \leq \bigwedge^{\mathbf{A}}\{\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) : y \in \widehat{W}\} < \alpha_{i+1}$ and so, by the definition of $T_{\Box}(x)$ (in any choice of universe for $\mathbf{A}$),

$$\widehat{V}(\Box\psi, x) = \bigvee^{\mathbf{A}}\{r \in \widehat{T}_{\Box}(x) : r \leq \bigwedge^{\mathbf{A}}\{\widehat{R}xy \to^{\mathbf{A}} \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = \alpha_i.$$

The third case, where $V(\Box\psi, x) \neq \alpha_i$, for all $i \leq m$, only occurs for $A = G_{\uparrow}$ and is straightforward. $\Box$

Corollary 8. $\mathsf{FK}(\mathbf{A})$ *and* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ *have the finite model property.*

Decidability now follows using the bounds on the sizes of models provided by Lemma 7 and restricting appropriately the values in $[0, 1]$ used in these models.

Lemma 9. *The validity problems of* $\mathsf{FK}(\mathbf{A})$ *and* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ *are decidable.*

Proof. Fix $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ and let $K = \ell(\varphi) + |\mathrm{C}_{\mathcal{L}}|$. By Lemma 7 and part of (b) of Lemma 6, to check the validity of φ in $\mathsf{FK}(\mathbf{A})$ or $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$, it suffices to check validity in $\mathsf{FK}(\mathbf{A})$-tree-models or $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-tree-models $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$

satisfying $|W| \leq |\Sigma(\varphi)|^{\ell(\varphi)} \leq (\ell(\varphi) + |\mathrm{C}_{\mathcal{L}}|)^{\ell(\varphi)} \leq K^K$. Moreover, we may restrict the values taken by R, V, $T_\Box$, and $T_\Diamond$ to be in a specified finite subset of $[0,1]$. In the case where $\mathbf{A}$ has universe $[0,1]$, each world $x \in W$ requires at most K values for the variables and K values for the members of $T_\Box(x) = T_\Diamond(x)$. Also, at most $(K^K)^2$ values are needed for the range of Rxy when $x, y \in W$. Hence at most $N = (2K)^{2K}$ values in $[0,1]$ are required. In particular, it is easy to see that we may restrict to values in the set $\{0, \frac{1}{N-1}, \ldots, \frac{N-2}{N-1}, 1\} \cup \mathrm{C}_{\mathcal{L}}^{\mathbf{A}}$. The cases of $G_\uparrow$ and $G_\downarrow$ are very similar. $\Box$

4. Equivalence of the Semantics

Let us again assume that $\mathbf{A}$ is an order-based algebra with universe $[0,1]$, $G_\uparrow$, or $G_\downarrow$. We devote this section to establishing that a formula is valid in $\mathsf{K}(\mathbf{A})$ or $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ if and only if it is valid in $\mathsf{FK}(\mathbf{A})$ or $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$, respectively. Decidability of the validity problems of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ then follows directly from Lemma 9.

We show first that given any finite fragment Σ and $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M}$, we can introduce suitable functions $T_\Box$ and $T_\Diamond$ to obtain an $\mathsf{FK}(\mathbf{A})$-model $\widehat{\mathfrak{M}}$ such that the evaluation of formulas in Σ is unchanged. It then follows using Lemma 3 that any $\mathsf{FK}(\mathbf{A})$-valid formula $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is also $\mathsf{K}(\mathbf{A})$-valid.

Lemma 10. *Let $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ be a finite fragment and $\mathfrak{M} = \langle W, R, V\rangle$ a $\mathsf{K}(\mathbf{A})$-model. Then there is an $\mathsf{FK}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle W, R, \widehat{V}, \widehat{T_\Box}, \widehat{T_\Diamond}\rangle$ such that $V(\varphi, x) = \widehat{V}(\varphi, x)$ for all $\varphi \in \Sigma$ and $x \in W$.*

Proof. First let $\widehat{V} = V$ and define for all $x \in W$:

(i) If A is $[0,1]$, then

$$\widehat{T_\Box}(x) = \widehat{T_\Diamond}(x) = V_x[\Sigma_\Box \cup \Sigma_\Diamond \cup \mathrm{C}_{\mathcal{L}}].$$

(ii) If A is $G_\downarrow$, then

$$\widehat{T_\Box}(x) = \{0, \min(V_x[\Sigma_\Box \cup \mathrm{C}_{\mathcal{L}}] \setminus \{0\}), \ldots, \tfrac{1}{2}, 1\} \quad \text{and} \quad \widehat{T_\Diamond}(x) = G_\downarrow.$$

(iii) If A is $G_\uparrow$, then

$$\widehat{T_\Box}(x) = G_\uparrow \quad \text{and} \quad \widehat{T_\Diamond}(x) = \{0, \tfrac{1}{2}, \ldots, \max(V_x[\Sigma_\Diamond \cup \mathrm{C}_{\mathcal{L}}] \setminus \{1\}), 1\}.$$

Then $\widehat{\mathfrak{M}} = \langle W, R, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ is an $\mathsf{FK}(\mathbf{A})$-model, and the fact that $V(\varphi, x) = \widehat{V}(\varphi, x)$ for all $\varphi \in \Sigma$ and $x \in W$ follows by induction on $\ell(\varphi)$. $\Box$

We now turn our attention to the other (much harder) direction: proving that any $\mathsf{K}(\mathbf{A})$-valid formula is also $\mathsf{FK}(\mathbf{A})$-valid. A central tool in the proof for the case where $A = [0, 1]$ is the following lemma which allows the "squeezing" of $\mathsf{K}(\mathbf{A})$-models so that the values of formulas are arbitrarily close to certain points.

Lemma 11. *Given $B \subseteq [0, 1]$, $0 = \alpha_0 < \ldots < \alpha_m = 1$, and $\beta_i \in (\alpha_i, \alpha_{i+1}) \setminus B$ for $i = 0, \ldots, m-1$:*

(a) *For each $k \in \mathbb{Z}^+$, there is a B-complete $\{\alpha_0, \ldots, \alpha_m\}$-order embedding $h_k \colon [0, 1] \to [0, 1]$ such that for $i = 0, \ldots, m-1$:*

$$h_k[[\alpha_i, \beta_i)] = [\alpha_i, \min(\alpha_i + \tfrac{1}{k}, \beta_i)) \quad \textit{and} \quad h_k[[\beta_i, \alpha_{i+1})] = [\beta_i, \alpha_{i+1}).$$

(b) *For each $k \in \mathbb{Z}^+$, there is a B-complete $\{\alpha_0, \ldots, \alpha_m\}$-order embedding $g_k \colon [0, 1] \to [0, 1]$ such that for $i = 0, \ldots, m-1$:*

$$g_k[(\alpha_i, \beta_i]] = (\alpha_i, \beta_i] \quad \textit{and} \quad g_k[(\beta_i, \alpha_{i+1}]] = (\max(\alpha_{i+1} - \tfrac{1}{k}, \beta_i), \alpha_{i+1}].$$

Proof. For (a), we can define h_k as the identity on $[\beta_i, \alpha_{i+1})$ and then linearly such that $h_k[[\alpha_i, \beta_i)] = [\alpha_i, \min(\alpha_i + \frac{1}{k}, \beta_i))$. Clearly h_k is an $\{\alpha_0, \ldots, \alpha_m\}$-order embedding. Moreover, it is easy to check that h_k is B-complete, using the fact that $\{\beta_0, \ldots, \beta_{m-1}\} \cap B = \emptyset$. (b) is very similar. $\Box$

We now provide the key construction of a $\mathsf{K}(\mathbf{A})$-tree-model taking the same values for formulas at its root as a given $\mathsf{FK}(\mathbf{A})$-tree-model. Note first that the original $\mathsf{FK}(\mathbf{A})$-model without the functions $T_\Box$ and $T_\Diamond$ cannot play this role in general; for example, in $[0, 1]$, the infimum or supremum required for calculating the value of a box-formula or diamond-formula at the root x might not be in the set $T_\Box(x) = T_\Diamond(x)$. This problem is resolved by taking infinitely many copies of the original $\mathsf{FK}(\mathbf{A})$-model (without the truth value functions) in such a way that the necessary parts of the intervals between members of $T_\Box(x)$ are "squeezed" closer to either their lower or upper bounds. The obtained infima and suprema will then coincide with the next smaller or larger member of $T_\Box(x)$: that is, the required values of the formulas at x in the original $\mathsf{FK}(\mathbf{A})$-model.

Lemma 12. *Suppose that $A = [0, 1]$. Let Σ be a finite fragment and $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ a finite $\mathsf{FK}(\mathbf{A})$-tree-model with root x. Then there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$. Moreover, if $\mathfrak{M}$ is crisp, then so is $\widehat{\mathfrak{M}}$.*

Proof. The lemma is proved by induction on $\mathrm{hg}(\mathfrak{M})$. The base case is immediate, fixing $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with $\widehat{W} = W = \{x\}$, $\widehat{R} = R$, and $\widehat{V} = V$. For the inductive step, given $y \in R^+[x]$, let $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ be the submodel of $\mathfrak{M}$ generated by $\{y\}$. Then $\mathfrak{M}_y$ is a finite $\mathsf{FK}(\mathbf{A})$-tree-model with root y, $\mathrm{hg}(\mathfrak{M}_y) < \mathrm{hg}(\mathfrak{M})$, and, by Lemma 6(a), $V_y(\varphi, z) = V(\varphi, z)$ for all $z \in W_y$ and $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$. So, by the induction hypothesis, there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle$ (crisp if $\mathfrak{M}$ is crisp) with root $\widehat{y}$ such that $\widehat{V}_y(\varphi, \widehat{y}) = V_y(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$.

We now use Lemma 11 to define infinitely many copies of $\widehat{\mathfrak{M}}_y$ such that at each copy, all the values of the formulas in Σ (and the accessibility relation) get "squeezed" closer and closer towards the next smaller (or next larger) element of $T_\Box(x) = T_\Diamond(x)$. Consider

$$T_\Box(x) = T_\Diamond(x) = \{\alpha_0, \ldots, \alpha_m\} \quad \text{with} \quad 0 = \alpha_0 < \ldots < \alpha_m = 1.$$

For any $y \in R^+[x]$ and $i = 0, \ldots, m-1$, let $b_i^\Box$ be the maximal element of the finite set

$$\{\alpha_i\} \cup (\{Rxy \to^{\mathbf{A}} V(\psi, y) : \Box\psi \in \Sigma_\Box\} \cap (\alpha_i, \alpha_{i+1}))$$

and let $b_i^\Diamond$ be the minimal element of the finite set

$$\{\alpha_{i+1}\} \cup (\{Rxy \wedge^{\mathbf{A}} V(\psi, y) : \Diamond\psi \in \Sigma_\Diamond\} \cap (\alpha_i, \alpha_{i+1})).$$

Because $\Omega(\widehat{\mathfrak{M}}_y, \Sigma)$ is countable, we may now choose for $i = 0, \ldots, m-1$:

$$\beta_i^\Box \in (b_i^\Box, \alpha_{i+1}) \setminus \Omega(\widehat{\mathfrak{M}}_y, \Sigma) \quad \text{and} \quad \beta_i^\Diamond \in (\alpha_i, b_i^\Diamond) \setminus \Omega(\widehat{\mathfrak{M}}_y, \Sigma).$$

By Lemma 11, we obtain for each even $k \in \mathbb{Z}^+$, an $\Omega(\widehat{\mathfrak{M}}_y, \Sigma)$-complete $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$-order embedding satisfying for $i = 0, \ldots, m-1$:

$$h_k[[\alpha_i, \beta_i^\Box)] = [\alpha_i, \min(\alpha_i + \tfrac{1}{k}, \beta_i^\Box)) \quad \text{and} \quad h_k[[\beta_i^\Box, \alpha_{i+1})] = [\beta_i^\Box, \alpha_{i+1}).$$

Similarly, for each odd $k \in \mathbb{Z}^+$, we obtain an $\Omega(\widehat{\mathfrak{M}}_y, \Sigma)$-complete $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$-order embedding satisfying for $i = 0, \ldots, m-1$:

$$h_k[(\alpha_i, \beta_i^\Diamond]] = (\alpha_i, \beta_i^\Diamond] \quad \text{and} \quad h_k[(\beta_i^\Diamond, \alpha_{i+1}]] = (\max(\beta_i^\Diamond, \alpha_{i+1} - \tfrac{1}{k}), \alpha_{i+1}].$$

We then define for each $k \in \mathbb{Z}^+$, a $\mathsf{K}(\mathbf{A})$-model $\widehat{\mathfrak{M}}_y^k = \langle \widehat{W}_y^k, \widehat{R}_y^k, \widehat{V}_y^k \rangle$ such that

(1) $\widehat{W}_y^k$ consists of a copy of $\widehat{W}_y$ extended with a new (root) world $\widehat{x}$, denoting the corresponding copy of $\widehat{x}_y \in \widehat{W}_y$ by $\widehat{x}_y^k$.

(2) $\widehat{R}_y^k \widehat{x}_y^k \widehat{z}_y^k = h_k(\widehat{R}_y \widehat{x}_y \widehat{z}_y)$ for all $\widehat{x}_y, \widehat{z}_y \in \widehat{W}_y$, $\widehat{R}_y^k \widehat{x}\widehat{y}^k = h_k(Rxy)$, and $\widehat{R}_y^k \widehat{x} w = 0$ for $w \neq \widehat{y}^k$.

(3) $\widehat{V}_y^k(p, \widehat{x}_y^k) = h_k(\widehat{V}_y(p, \widehat{x}_y))$ for $\widehat{x}_y \in \widehat{W}_y$, and $\widehat{V}_y^k(p, \widehat{x}) = V(p, x)$.

Because h_k is an $\Omega(\widehat{\mathfrak{M}}_y, \Sigma)$-complete $\mathrm{C}_{\mathcal{L}}^{\mathbf{A}}$-order embedding, applying Lemmas 1 and 2, it holds that for all $\varphi \in \Sigma$:

$$\widehat{V}_y^k(\varphi, \widehat{y}^k) = h_k(\widehat{V}_y(\varphi, \widehat{y})).$$

Observe, moreover, that for each $\Box\psi \in \Sigma_\Box$ and $k \in \mathbb{Z}^+$:

$$\begin{array}{rcl} \widehat{R}_y^k \widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}_y^k(\psi, \widehat{y}^k) & = & h_k(Rxy) \to^{\mathbf{A}} h_k(\widehat{V}_y(\psi, \widehat{y})) \\ & = & h_k(Rxy \to^{\mathbf{A}} \widehat{V}_y(\psi, \widehat{y})) \\ & = & h_k(Rxy \to^{\mathbf{A}} V(\psi, y)). \end{array}$$

Hence, either $Rxy \to^{\mathbf{A}} V(\psi, y) = 1$ and so also $\widehat{R}_y^k \widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}_y^k(\psi, \widehat{y}^k) = 1$, or for some $i \in \{0, \ldots, m-1\}$ and suitably large even $k \in \mathbb{Z}^+$:

$$\widehat{R}_y^k \widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}_y^k(\psi, \widehat{y}^k) \in [\alpha_i, \alpha_i + 1/k).$$

We now define the $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ (with root $\widehat{x}$, which occurs in each $\widehat{W}_y^k$) by

$$\begin{array}{rcl} \widehat{W} & = & \bigcup_{y \in R^+[x]} \bigcup_{k \in \mathbb{Z}^+} \widehat{W}_y^k \\ \widehat{R}wz & = & \begin{cases} \widehat{R}_y^k wz & \text{if } w, z \in \widehat{W}_y^k \text{ for some } y \in R^+[x] \text{ and } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases} \\ \widehat{V}(p, z) & = & \widehat{V}_y^k(p, z) \quad \text{if } z \in \widehat{W}_y^k \text{ for some } y \in R^+[x] \text{ and } k \in \mathbb{Z}^+. \end{array}$$

If $\mathfrak{M}$ is crisp, then for all $y \in R^+[x]$, $\widehat{\mathfrak{M}}_y$ is crisp and so also are $\widehat{\mathfrak{M}}_y^k$ for all $k \in \mathbb{Z}^+$. Hence, by construction, $\widehat{\mathfrak{M}}$ is crisp. Moreover, as there are only finitely many different countable $\widehat{\mathfrak{M}}_y$, and we only take countably many copies of each, $\widehat{\mathfrak{M}}$ is also countable.

Observe now that for each $\widehat{y}^k \in \widehat{W}$, we have that $\widehat{\mathfrak{M}}_y^k$ is the submodel of $\widehat{\mathfrak{M}}$ generated by $\{\widehat{y}^k\}$. Hence by Lemma 2, for all $\varphi \in \Sigma$ and $\widehat{y}^k \in \widehat{R}^+[\widehat{x}]$,

$$\widehat{V}(\varphi, \widehat{y}^k) = \widehat{V}_y^k(\varphi, \widehat{y}^k) = h_k(\widehat{V}_y(\varphi, \widehat{y})) = h_k(V_y(\varphi, y)) = h_k(V(\varphi, y)).$$

Finally, we prove that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$, proceeding by induction on $\ell(\varphi)$. The base case $\ell(\varphi) = 1$ follows directly from the definition of $\widehat{V}$ (noting that $\mathrm{C}_{\mathcal{L}}^{\mathbf{A}} \subseteq T_\Box(x) = \{\alpha_0, \ldots, \alpha_m\}$). For the inductive step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case $\varphi = \Box\psi$ (a formula in $\Sigma_\Box$), the case $\varphi = \Diamond\psi$ being very similar. Recall that

$$V(\Box\psi, x) = \bigvee\nolimits^{\mathbf{A}}\{r \in T_\Box(x) : r \leq \bigwedge\nolimits^{\mathbf{A}}\{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\}\}.$$

Hence $V(\Box\psi, x) = \alpha_i$ for some $i \in \{0, \ldots, m\}$. If $i = m$, then for all $y \in W$:

$$Rxy \to^{\mathbf{A}} V(\psi, y) = 1.$$

But then also for each $y \in W$ and $k \in \mathbb{Z}^+$:

$$\widehat{R}\widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{y}^k) = \widehat{R}_y^k\widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}_y^k(\psi, \widehat{y}^k) = 1.$$

So $\widehat{V}(\Box\psi, \widehat{x}) = 1$ as required.

Suppose now that $i < m$. Then for $z \in W$, we have $Rxz \to^{\mathbf{A}} V(\psi, z) \geq \alpha_i$. Hence also, by construction, for all $\widehat{z} \in \widehat{W}$,

$$\widehat{R}\widehat{x}\widehat{z} \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{z}) \geq \alpha_i.$$

Moreover, for some $y \in W$,

$$Rxy \to^{\mathbf{A}} V(\psi, y) \in [\alpha_i, \alpha_{i+1}),$$

and it follows that

$$\begin{aligned} \alpha_i &\leq \bigwedge\nolimits^{\mathbf{A}}\{\widehat{R}\widehat{x}\widehat{z} \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{z}) : \widehat{z} \in \widehat{W}\} \\ &\leq \bigwedge\nolimits^{\mathbf{A}}\{\widehat{R}_y^k\widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}_y^k(\psi, \widehat{y}^k) : k \in \mathbb{Z}^+\} \\ &\leq \bigwedge\nolimits^{\mathbf{A}}\{\alpha_i + \tfrac{1}{k} : k \in \mathbb{Z}^+\} \\ &= \alpha_i. \end{aligned}$$

So $\widehat{V}(\Box\psi, \widehat{x}) = \alpha_i = V(\Box\psi, x)$ as required. □

For $A = G_\downarrow$ and $A = G_\uparrow$, the situation is a little less complicated, as the only non-isolated point is either 0 or 1. In particular, for $G_\downarrow$, when treating an $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ with root x and $T_\Box(x) = \{0, \frac{1}{m}, \ldots, \frac{1}{2}, 1\}$, we only "squeeze" values in the interval $(0, \frac{1}{m})$ towards 0.

Lemma 13. *Suppose that $A = G_\downarrow$ or $A = G_\uparrow$. Let Σ be a finite fragment and $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ a finite $\mathsf{FK}(\mathbf{A})$-tree-model with root x. Then there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}$, such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$. Moreover, if $\mathfrak{M}$ is crisp, then so is $\widehat{\mathfrak{M}}$.*

Proof. Let us consider $A = G_\downarrow$, the case $A = G_\uparrow$ being very similar. We follow closely the proof of Lemma 12 and proceed by induction on $\mathrm{hg}(\mathfrak{M})$. The base case is trivial. For the inductive step, we again define for each $y \in R^+[x]$, the $\mathsf{FK}(\mathbf{A})$-model $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ as the submodel of $\mathfrak{M}$ generated by $\{y\}$ and observe that by the induction hypothesis, for each $y \in R^+[x]$, there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle$ (crisp if $\mathfrak{M}$ is crisp) with root $\widehat{y}$ such that $\widehat{V}_y(\varphi, \widehat{y}) = V_y(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$.

Suppose that $T_\Box(x) = \{0, \frac{1}{m}, \ldots, \frac{1}{2}, 1\}$. Similarly to the proof of Lemma 12, we define infinitely many copies of our models $\widehat{\mathfrak{M}}_y$, but in this case, values of formulas (and the accessibility relation) smaller than $\frac{1}{m}$ are "squeezed" closer and closer towards 0. We define $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$-order embeddings $h_k \colon G_\downarrow \to G_\downarrow$ for $k \in \mathbb{Z}^+$ such that

$$h_k(\tfrac{1}{i}) = \begin{cases} \frac{1}{i} & \text{if } 1 \leq i \leq m \\ \frac{1}{i+k} & \text{otherwise.} \end{cases}$$

As 0 is the only non-isolated element of $G_\downarrow$, each h_k is $G_\downarrow$-complete.

We then define for each $y \in R^+[x]$ and $k \in \mathbb{Z}^+$, a $\mathsf{K}(\mathbf{A})$-model $\widehat{\mathfrak{M}}^k_y = \langle \widehat{W}^k_y, \widehat{R}^k_y, \widehat{V}^k_y \rangle$ as in the proof of Lemma 12. That is, let $\widehat{W}^k_y$ be a copy of $\widehat{W}_y$ extended with (a new root) $\widehat{x}$. For all $\widehat{x}_y, \widehat{z}_y \in \widehat{W}_y$, we define $\widehat{R}^k_y \widehat{x}^k_y \widehat{z}^k_y = h_k(\widehat{R}_y \widehat{x}_y \widehat{z}_y)$ and $\widehat{R}^k_y \widehat{x} \widehat{z}^k_y = h_k(Rxy)$ if $\widehat{z}^k_y = \widehat{y}^k$ (is a copy of the root $\widehat{y}$), 0 otherwise. Also, let $\widehat{V}^k_y(p, \widehat{x}^k_y) = h_k(\widehat{V}_y(p, \widehat{x}_y))$ for all $\widehat{x}_y \in \widehat{W}_y$, and $\widehat{V}^k_y(p, \widehat{x}) = V(p, x)$.

Observe that we can again apply Lemma 1 so that for all $y \in R^+[x]$, $k \in \mathbb{Z}^+$, and $\varphi \in \Sigma$, we obtain $\widehat{V}^k_y(\varphi, \widehat{y}^k) = h_k(\widehat{V}_y(\varphi, \widehat{y}))$. Hence

$$\widehat{R}^k_y \widehat{x} \widehat{y}^k \to^{\mathbf{A}} \widehat{V}^k_y(\varphi, \widehat{y}^k) = h_k(Rxy \to^{\mathbf{A}} V(\varphi, y)).$$

In particular, if $Rxy \to^{\mathbf{A}} V(\varphi, y) < \frac{1}{m}$, then for each $k \in \mathbb{Z}^+$

$$\widehat{R}^k_y \widehat{x} \widehat{y}^k \to^{\mathbf{A}} \widehat{V}^k_y(\psi, \widehat{y}^k) < \tfrac{1}{m+k}.$$

We further define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ by

$$\begin{array}{rcl} \widehat{W} & = & \bigcup_{y \in R^+[x]} \bigcup_{k \in \mathbb{Z}^+} \widehat{W}_y^k \\ \widehat{R}wz & = & \begin{cases} \widehat{R}_y^k wz & \text{if } w, z \in \widehat{W}_y^k \text{ for some } y \in R^+[x] \text{ and } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases} \\ \widehat{V}(p,z) & = & \widehat{V}_y^k(p,z) \quad \text{if } z \in \widehat{W}_y^k \text{ for some } y \in R^+[x] \text{ and } k \in \mathbb{Z}^+. \end{array}$$

Again, we note that crispness is preserved, $\widehat{W}$ is countable, and that $\widehat{V}(\varphi, \widehat{y}^k) = \widehat{V}_y^k(\varphi, \widehat{y}^k) = h_k(\widehat{V}_y(\varphi, \widehat{y})) = h_k(V_y(\varphi, y)) = h_k(V(\varphi, y))$ for all $\varphi \in \Sigma$ and $\widehat{y}^k \in \widehat{R}^+[\widehat{x}]$. It then follows by induction on $\ell(\varphi)$ that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$ (similarly to the proof of Lemma 12). Recall that

$$V(\Box\psi, x) = \bigvee^{\mathbf{A}} \{r \in T_\Box(x) : r \leq \bigwedge^{\mathbf{A}} \{Rxy \to^{\mathbf{A}} V(\psi, y) : y \in W\}\}.$$

Hence $V(\Box\psi, x) = \frac{1}{i}$ for some $i \in \{1, \ldots, m\}$ or $V(\Box\psi, x) = 0$. If the former, then it is the case for all $y \in W$ that $Rxy \to^{\mathbf{A}} V(\psi, y) \geq \frac{1}{i}$ and there is a $y_0 \in W$ such that

$$Rxy_0 \to^{\mathbf{A}} V(\psi, y_0) = \tfrac{1}{i}.$$

But then also for each $y \in R^+[x]$ and all $k \in \mathbb{Z}^+$,

$$\widehat{R}\widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{y}^k) \geq \tfrac{1}{i} \quad \text{and} \quad \widehat{R}\widehat{x}\widehat{y}_0^k \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{y}_0^k) = \tfrac{1}{i}.$$

So $\widehat{V}(\Box\psi, \widehat{x}) = \frac{1}{i}$ as required.

Suppose now that $V(\Box\psi, x) = 0$. Then for some $y \in W$,

$$Rxy \to^{\mathbf{A}} V(\psi, y) < \tfrac{1}{m},$$

and it follows that

$$\begin{array}{rcl} \widehat{V}(\Box\psi, \widehat{x}) & = & \bigwedge^{\mathbf{A}} \{\widehat{R}\widehat{x}\widehat{z} \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{z}) : \widehat{z} \in \widehat{W}\} \\ & \leq & \bigwedge^{\mathbf{A}} \{\widehat{R}\widehat{x}\widehat{y}^k \to^{\mathbf{A}} \widehat{V}(\psi, \widehat{y}^k) : k \in \mathbb{Z}^+\} \\ & \leq & \bigwedge^{\mathbf{A}} \{\frac{1}{m+k} : k \in \mathbb{Z}^+\} \\ & = & 0. \end{array}$$

So $\widehat{V}(\Box\psi, \widehat{x}) = 0 = V(\Box\psi, x)$ as required. $\Box$

We obtain the following equivalence result.

Theorem 14. *Let* $\mathbf{A}$ *be an order-based algebra with universe* $[0, 1]$*,* $G_\uparrow$*, or* $G_\downarrow$*. Then for all* $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$*:*

(a) $\models_{\mathsf{K}(\mathbf{A})} \varphi$ *if and only if* $\models_{\mathsf{FK}(\mathbf{A})} \varphi$.

(b) $\models_{\mathsf{K}(\mathbf{A})^{\mathsf{C}}} \varphi$ *if and only if* $\models_{\mathsf{FK}(\mathbf{A})^{\mathsf{C}}} \varphi$.

Proof. For (a), suppose first that $\not\models_{\mathsf{FK}(\mathbf{A})} \varphi$. By Lemmas 6 and 7, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ with root x such that $V(\varphi, x) < 1$. By Lemma 12 or 13, we obtain a $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x) < 1$. So $\not\models_{\mathsf{K}(\mathbf{A})} \varphi$. Conversely, suppose that $\not\models_{\mathsf{K}(\mathbf{A})} \varphi$. Then there is a $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V \rangle$ and $x \in W$ such that $V(\varphi, x) < 1$. By Lemma 10, we obtain an $\mathsf{FK}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle W, R, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ such that $\widehat{V}(\varphi, x) < 1$.

The proof of (b) is very similar using the fact that Lemmas 6, 7, 12, 13, and 10 preserve crispness. □

Combining Theorem 14 and Lemma 9, we obtain the main result of this paper.

Theorem 15. *Let* $\mathbf{A}$ *be an order-based algebra with universe* $[0, 1]$*,* $G_\uparrow$*, or* $G_\downarrow$*. Then the validity problems of* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ *are decidable.*

5. Order-Based Crisp S5 Logics

As in the classical setting, further many-valued modal logics may be defined as logics of particular classes of $\mathsf{K}(\mathbf{A})$-models for a given order-based algebra $\mathbf{A}$. In this section, we focus only on proving decidability and co-NP-inclusion for crisp "S5" order-based logics that may be viewed also as one-variable fragments of order-based first-order logics.

We define an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-model* to be a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V, R \rangle$ such that R is an equivalence relation. We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case just write $\mathfrak{M} = \langle W, V \rangle$, noting that the clauses for $\Box$ and $\Diamond$ simplify to

$$
\begin{aligned}
V(\Box\varphi, x) &= \textstyle\bigwedge^{\mathbf{A}} \{V(\varphi, y) : y \in W\} \\
V(\Diamond\varphi, x) &= \textstyle\bigvee^{\mathbf{A}} \{V(\varphi, y) : y \in W\}.
\end{aligned}
$$

The following lemma is an immediate corollary of Lemma 2 and the fact that the submodel of an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model generated by one world is universal.

Lemma 16. $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ *if and only if* φ *is valid in all universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-models.*

It follows that each many-valued modal logic $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ may be viewed as the one-variable fragment of a corresponding order-based first-order logic. Rather than define this first-order logic and then restrict to its one-variable fragment, let us simply note that the first-order translation of $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ is obtained by replacing each propositional variable p with the predicate $p(x)$, $\Box$ with $\forall x$, and $\Diamond$ with $\exists x$. In particular, when $\mathbf{A}$ is the standard Gödel algebra $\mathbf{G}$, we obtain the Gödel modal logic $\mathsf{GS5}^{\mathsf{C}}$ corresponding to the one-variable fragment of first-order Gödel logic (see, e.g., [1, 16]). The logic $\mathsf{GS5}^{\mathsf{C}}$ can be axiomatized as an extension of the intuitionistic modal logic MIPC studied in [6, 26] with the prelinearity axiom schema and $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ [9]. (It is also shown in [9] that the logic $\mathsf{GS5}$ based on non-crisp frames may be axiomatized as MIPC with just prelinearity, and decidability of the validity problem follows from the finite model property for the semantics with two relations [2].) One-variable fragments of other first-order Gödel logics are obtained when $\mathbf{A}$ is the algebra $\mathbf{G}_{\uparrow}$ or $\mathbf{G}_{\downarrow}$ (see [1]).

The infinite $\mathsf{K}(\mathbf{A})$-model defined in the proof of Theorem 5 for the formula $\Box\neg\neg p \to \neg\neg\Box p$ is a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model. Hence, if the universe of $\mathbf{A}$ is $[0,1]$ or $G_{\downarrow}$ and the language includes $\to$, then $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ does not have the finite model property. Also, as in Theorem 4, the logic $\mathsf{S5}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ has the finite model property, but not if Δ is added to the language. We will prove decidability for all these cases using again a new equivalent semantics.

Let us assume that the universe of an order-based algebra $\mathbf{A}$ is $[0,1]$, $G_{\downarrow}$, or $G_{\uparrow}$. We define an $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-model* as an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$ such that $\langle W, R, V\rangle$ is an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model, and $T_{\Box}(x) = T_{\Box}(y)$ and $T_{\Diamond}(x) = T_{\Diamond}(y)$ whenever Rxy. We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case write $\mathfrak{M} = \langle W, V, T_{\Box}, T_{\Diamond}\rangle$, where $T_{\Box}$ and $T_{\Diamond}$ may now be understood as fixed subsets of A, and the clauses for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\nolimits^{\mathbf{A}}\{r \in T_{\Box} : r \leq \bigwedge\nolimits^{\mathbf{A}}\{V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\nolimits^{\mathbf{A}}\{r \in T_{\Diamond} : r \geq \bigvee\nolimits^{\mathbf{A}}\{V(\varphi, y) : y \in W\}\}. \end{aligned}$$

Note in particular that in universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models and $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models, the truth values of box-formulas and diamond-formulas are independent of the world.

We now show that $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity is equivalent to validity in finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models, following quite closely the corresponding proofs from previous sections.

Lemma 17. *Let* $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ *be a finite fragment,* $\mathfrak{M} = \langle W, V\rangle$ *a universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-model, and* $x \in W$*. Then there is a finite universal* $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-model*

$\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ *with* $\langle \widehat{W}, \widehat{V} \rangle \subseteq \langle W, V \rangle$, $x \in \widehat{W}$, *and* $|\widehat{W}| \leq |\Sigma|$ *such that* $\widehat{V}(\varphi, y) = V(\varphi, y)$ *for all* $\varphi \in \Sigma$ *and* $y \in \widehat{W}$.

Proof. Given Σ, $\mathfrak{M}$, and x as stated above, we define $\widehat{T}_\Box \subseteq [0,1]$ and $\widehat{T}_\Diamond \subseteq [0,1]$ as follows:

(i) If A is $[0,1]$, then $\widehat{T}_\Box = \widehat{T}_\Diamond = V_x[\Sigma_\Box \cup \Sigma_\Diamond \cup \mathrm{C}_\mathcal{L}]$.

(ii) If A is $G_\downarrow$, then $\widehat{T}_\Diamond = G_\downarrow$ and $\widehat{T}_\Box = \{0, \min(V_x[\Sigma_\Box \cup \mathrm{C}_\mathcal{L}] \setminus \{0\}), \ldots, \frac{1}{2}, 1\}$.

(iii) If A is $G_\uparrow$, then $\widehat{T}_\Box = G_\uparrow$ and $\widehat{T}_\Diamond = \{0, \frac{1}{2}, \ldots, \max(V_x[\Sigma_\Diamond \cup \mathrm{C}_\mathcal{L}] \setminus \{1\}), 1\}$.

For each $\Box\psi \in \Sigma_\Box$, we choose a $y_{\Box\psi} \in W$ such that $V(\Box\psi, x) = \bigvee^{\mathbf{A}}\{r \in \widehat{T}_\Box : r \leq V(\psi, y_{\Box\psi})\}$. Similarly, for each $\Diamond\psi \in \Sigma_\Diamond$, we choose a $y_{\Diamond\psi} \in W$ such that $V(\Diamond\psi, x) = \bigwedge^{\mathbf{A}}\{r \in \widehat{T}_\Diamond : r \geq V(\psi, y_{\Box\psi})\}$. We let $\widehat{W} = \{x\} \cup \{y_\varphi \in W : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\} \subseteq W$ and observe that $|\widehat{W}| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 1 \leq |\Sigma|$. Now let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ where $\widehat{V}$ is V restricted to $\widehat{W}$. It follows by induction on $\ell(\varphi)$ that $\widehat{V}(\varphi, y) = V(\varphi, y)$ for all $y \in \widehat{W}$ and $\varphi \in \Sigma$. □

Lemma 18. *Let* $\Sigma \subseteq \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ *be a finite fragment and* $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ *a finite universal* $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-model. Then there is a universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-model* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ *where* $W \subseteq \widehat{W}$ *such that* $\widehat{V}(\varphi, x) = V(\varphi, x)$ *for all* $\varphi \in \Sigma$ *and* $x \in W$.

Proof. Given a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M}$, we construct our universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\widehat{\mathfrak{M}}$ directly by taking infinitely many copies of $\mathfrak{M}$. Let us restrict our attention to the case $A = [0,1]$, noting that the cases $A = G_\downarrow$ and $A = G_\uparrow$ are very similar (where the family $\{h_k\}_{k \in \mathbb{Z}^+}$ of embeddings is defined as in the proof of Lemma 13).

Consider $T_\Box = T_\Diamond = \{\alpha_0, \ldots, \alpha_m\}$ with $0 = \alpha_0 < \ldots < \alpha_m = 1$ and define a family of $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$-order embeddings $\{h_k\}_{k \in \mathbb{Z}^+}$ such that for each even $k \in \mathbb{Z}^+$ and $i \in \{0, \ldots, m-1\}$,

$$h_k[[\alpha_i, \alpha_{i+1})] = [\alpha_i, \min(\alpha_i + \tfrac{1}{k}, \alpha_{i+1}))$$

and for every odd $k \in \mathbb{Z}^+$ and $i \in \{0 \ldots m-1\}$,

$$h_k[(\alpha_i, \alpha_{i+1}]] = (\max(\alpha_i, \alpha_{i+1} - \tfrac{1}{k}), \alpha_{i+1}].$$

Note that because W is finite, we do not have to worry about infinite meets and joins here and thus do not need to define $\{h_k\}_{k \in \mathbb{Z}^+}$ as carefully as in the proof of

Lemma 12. Let h_0 be the identity on $[0,1]$, $\widehat{W}_0 = W$, and for each $k \in \mathbb{Z}^+$, let $\widehat{W}_k$ be a copy of W with a distinct copy $\widehat{x}_k$ for each $x \in W$; also let $\widehat{x}_0 = x$ for each $x \in W$. We define the universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ where

$$\widehat{W} = \bigcup_{k \in \mathbb{N}} \widehat{W}_k \quad \text{and} \quad \widehat{V}(p, \widehat{x}_k) = h_k(V(p,x)) \text{ for all } x \in W \text{ and } k \in \mathbb{N}.$$

It suffices now to prove that for all $\varphi \in \Sigma$, $x \in W$, and $k \in \mathbb{N}$,

$$\widehat{V}(\varphi, \widehat{x}_k) = h_k(V(\varphi, x)),$$

proceeding by induction on $\ell(\varphi)$. The base case follows by definition, while for the non-modal connectives, the argument is the same as in the proof of Lemma 1. Consider $\varphi = \Diamond\psi$, noting that the case $\varphi = \Box\psi$ is very similar. Fix $x \in W$ and $k \in \mathbb{N}$. Suppose first that $V(\Diamond\psi, x) = 0$. Then $V(\psi, y) = 0$ for all $y \in W$ and hence $\widehat{V}(\psi, \widehat{y}_n) = h_n(V(\psi, y)) = 0$ for all $y \in W$ and $n \in \mathbb{N}$. So $\widehat{V}(\Diamond\psi, \widehat{x}_k) = 0$ as required. Now suppose that $V(\Diamond\psi, x) = \alpha_i$ for some $i \in \{1, \ldots, m\}$. Then $\bigvee^{\mathbf{A}}\{V(\psi, y) : y \in W\} \in (\alpha_{i-1}, \alpha_i]$ and thus for some $y \in W$, we have $V(\psi, y) \in (\alpha_{i-1}, \alpha_i]$. Therefore if follows for any sufficiently large odd $n \in \mathbb{Z}^+$ that $h_n(V(\psi, y)) \in (\alpha_i - \frac{1}{n}, \alpha_i]$. So, using the induction hypothesis:

$$\begin{aligned}
\widehat{V}(\Diamond\psi, \widehat{x}_k) &= \textstyle\bigvee^{\mathbf{A}}\{\widehat{V}(\psi, \widehat{y}_n) : y \in W, n \in \mathbb{N}\} \\
&= \textstyle\bigvee^{\mathbf{A}}\{h_n(V(\psi, y)) : y \in W, n \in \mathbb{N}\} \\
&= \textstyle\bigvee^{\mathbf{A}}\{\alpha_i - \frac{1}{n} : n \in \mathbb{Z}^+\} \\
&= \alpha_i.
\end{aligned}$$

So $\widehat{V}(\Diamond\psi, \widehat{x}_k) = \alpha_i = h_k(\alpha_i)$ as required. $\square$

Combining Lemmas 16, 17, and 18, we obtain the following equivalence.

Theorem 19. *Let* $\mathbf{A}$ *be an order-based algebra with universe* $[0,1]$, $G_{\uparrow}$, *or* $G_{\downarrow}$. *Then* $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ *if and only if* φ *is valid in all finite universal* $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-models.*

The desired decidability and complexity result is then obtained by considering the number of truth values needed to check validity of formulas in finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models.

Theorem 20. *Let* $\mathbf{A}$ *be an order-based algebra with universe* $[0,1]$, $G_{\uparrow}$, *or* $G_{\downarrow}$. *Then the validity problem of* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ *is in co-NP.*

Proof. Consider $\varphi \in \mathrm{Fm}^{\mathcal{L}}_{\Box\Diamond}$ and let $K = \ell(\varphi) + |\mathrm{C}_{\mathcal{L}}|$. To check if φ is not $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-valid, it suffices, by Lemmas 17 and 18, to check if φ is not valid in a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ with $|W| = K$. Observe that in such a model, a maximum of K^2 distinct values in A are required for the values of the variables in $\mathrm{Var}(\varphi)$ and values in $T_\Box$ and $T_\Diamond$ (when these are finite). Moreover, we may assume that these values are in a fixed finite set. In particular, for $A = [0,1]$, we could use the set $\{0, \frac{1}{K^2-1}, \ldots, \frac{K^2-2}{K^2-1}, 1\} \cup \mathrm{C}_{\mathcal{L}}$. Guessing these values and $x \in W$ non-deterministically, then computing the value of $V(\varphi, x)$ can be achieved in time polynomial in K. So the validity problem of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ is in co-NP. $\Box$

For crisp Gödel S5 logics and the corresponding one-variable fragments of first-order Gödel logics, the validity problem is co-NP-hard (this is already true for the propositional case, see [16]) and hence we may conclude the following.

Theorem 21. *The validity problems of the one-variable fragment of (standard) first-order Gödel logic and the one-variable fragments of the first-order Gödel logics based on the algebras* $\mathbf{G}_\uparrow$ *and* $\mathbf{G}_\downarrow$ *are co-NP-complete.*

6. Concluding Remarks

In this paper, we have established the decidability of the validity problem for certain "order-based" modal logics, including the Gödel modal logics investigated in [5, 8, 9, 23]. We have also established decidability and co-NP-inclusion for the validity problem of "crisp S5" versions of these logics corresponding to one-variable fragments of first-order logics. In particular, we have answered positively the open problem of the decidability (indeed, co-NP-completeness) of the validity problem for the one-variable fragment of first-order Gödel logic. There remain, however, a number of significant questions, notably:

- *What is the complexity of validity in minimal order-based modal logics?* The box and diamond fragments of the Gödel modal logics GK and GK^{C} are known to be PSPACE-complete (see [23]) and we conjecture that this is also true of the full logics and a general class of many-valued modal logics.

- *Are order-based multi-modal logics also decidable?* This question is of particular interest as many-valued description logics (see, e.g., [4, 18, 29]) may be viewed as many-valued multi-modal logics. The challenge in this case is to extend the new semantics to a multi-modal setting.

- *Is the new semantics suitable for other classes of order-based modal logics?* It seems to be feasible to establish decidability for the validity problem of $\mathsf{K}(\mathbf{A})$, $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$, and $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ where $\mathbf{A}$ is *any* order-based algebra. Proving this, however, will involve a careful consideration of the order types of the underlying chains. Extending the approach to frames satisfying conditions such as reflexivity, symmetry, and transitivity may require new ideas.

- *Is validity in the two-variable fragment of first-order Gödel logic decidable?* Notably, validity in the two-variable fragment of first-order classical logic (indeed, any first-order tabular intermediate logic) is decidable [24], while the same fragment of first-order intuitionistic logic is undecidable [21].

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