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Proof theory

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Research Report 2011-07

5.10.2011

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Chapter I: Proof Theory

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1 Introduction

As clearly demonstrated in other chapters of this handbook, fuzzy logics are motivated first and foremost by semantic considerations: in particular, by the goal of representing and reasoning about truth degrees. However, as logics, they evidently deal also with the notion of *proof*. This much is apparent in Hilbert-style axiom systems, which (with enough patience) will churn out each theorem of the logic using a few simple rules and a list of axiom schema. But while Hilbert systems are a convenient formalism for presenting logics corresponding to classes of algebras, they are not so flexible when it comes to searching for, analyzing, and reasoning about proofs. At each step in a Hilbert system proof, the next axiom or instance of a rule like modus ponens should be guessed. Much better are proof systems with more restrictions on how to proceed: ideally, systems where proofs are *analytic*, built from the raw material (subformulas) of the formula to be proved. Such systems and their applications will be the subject of this chapter.

Surprisingly perhaps, most fuzzy logics investigated in the literature have a very natural proof-theoretic formulation. They occur (alongside intuitionistic logic, relevant logics, linear logic, Lambek calculus, etc.) as substructural logics in the framework of *Gentzen systems*. Typically, such systems gain flexibility by not dealing directly with formulas, but rather with *sequents*: ordered pairs of sequences (or sets or multisets) of formulas. So-called sequent calculi provide a natural home for a range of logics from linguistics, philosophy, computer science, and mathematics, as well as corresponding directly to some well studied classes of algebras. However, for fuzzy logics, sequents are not quite enough. A step up in complexity is required to *hypersequents*: multisets of sequents interpreted as disjunctions. Proof systems for many fuzzy logics are then obtained simply by transferring sequent calculi to the hypersequent level and adding an extra rule (or two). For example, a hypersequent calculus for Gödel logic is obtained by allowing hypersequent contexts in Gentzen's sequent calculus for intuitionistic logic and adding a rule permitting "communication" between sequents.

The goal of this chapter, however, is not only to show that fuzzy logics can be presented proof-theoretically, but also to show why proof theory matters for this field. What can be done with analytic proof systems that cannot be achieved with a purely algebraic approach? For many substructural logics, a standard answer would be that these systems are essential for establishing algorithmic properties such as decidability, complexity, and interpolation. In particular, the only known proofs of decidability for the full Lambek calculus (which extends even to the first-order level) or, equivalently, the equational theory of (pointed) residuated lattices, make use of Gentzen systems. For hypersequents, the situation is not so straightforward, however. Decidability and complexity results can be obtained in certain cases, but sometimes the mere existence of an analytic calculus is of no help whatsoever. So why care about developing proof theory for fuzzy logics?

This chapter concentrates on two important applications. First, proof theory provides a novel approach to tackling the central problem in mathematical fuzzy logic of *standard completeness*: showing that a logic is complete with respect to the intended fuzzy semantics, or, equivalently, showing that a variety of algebras is generated by certain distinguished members. Roughly, the idea is to add a special "density" rule to a logic that guarantees standard completeness and then to show proof-theoretically that it can be eliminated. For example, this method provides the only known proof of the standard completeness of uninorm logic, or, equivalently, the generation of the variety of semilinear bounded pointed commutative residuated lattices by its members with lattice reduct $[0, 1]$. The second key application of proof theory considered here is the extension of propositional fuzzy logics to the first-order level, a step that is somewhat problematic algebraically but completely natural for Gentzen systems. The proof-theoretic presentation facilitates investigation of key topics such as Herbrand's theorem and Skolemization, as well as providing a means for tackling (fragments of) non recursively axiomatizable cases such as first-order Łukasiewicz logic. Also discussed are the (hitherto proof-theoretically rather underdeveloped) topics of propositional fuzzy logics extended with modalities and propositional quantifiers.

The focus in this chapter will be on concepts and ideas, avoiding the temptation to generalize or over-complicate, but important techniques and landmark results will be presented in detail. Proofs, or at least proof sketches, are provided except in cases bearing significant similarity to preceding results or involving considerable technicalities. Nevertheless, due to size limitations, several interesting topics have been omitted. In particular, only commutative infinite-valued logics are considered here, avoiding well-developed but rather distinct approaches to finite-valued logics and similar but more intricate presentations of noncommutative fuzzy logics. We also limit our attention to Gentzen systems. Other proof frameworks such as natural deduction, tableaux, resolution, and display logic, to name just a few, each have their own distinct advantages and disadvantages. However, not only do Gentzen systems appear to be the most natural and useful formalism in the context of fuzzy logics, but also ideas and results from this domain can typically be transferred to other frameworks. For further details, we recommend consulting the monograph [53], and the historical remarks collected at the end of the chapter.

2 A Gentzen systems primer

In this section, we introduce and begin to explore some of the main ideas underlying Gentzen systems, in particular, how and when they can be developed and what they might be good for. We focus on three case studies – lattices, classical logic, and substructural logics – to showcase the diversity of the framework, and end with the obvious question: how can we extend these methods to fuzzy logics?

2.1 Rules, systems, derivations

Let us start with some general definitions that will serve us well in what follows. First, note that the objects of proof systems can be more complicated entities than formulas: equations, sequents, hypersequents, sequents of sequents, and so on. We therefore assume only that these objects constitute a set Φ of *structures*. An *(inference) rule* (r) for Φ is then a set of ordered pairs $\langle \Psi, a \rangle$ called *inferences* where $\Psi \subseteq \Phi$ is a finite (possibly empty) set of *premises* and $a \in \Phi$ is the *conclusion*, typically written for $\Psi = \{a_1, \ldots, a_n\}$ as $a_1, \ldots, a_n \;/\; a$ or $\frac{a_1 \,\cdots\, a_n}{a}$. An inference with no premises is called an *axiom*. Usually, axioms and rules are defined via schema that use meta-variables to stand for arbitrary members of Φ or sets based on Φ such as multisets of members of Φ.

A *proof system* C is an ordered pair $\langle \Phi, R \rangle$ consisting of a set Φ of structures and a set R of rules for Φ. A C*-derivation* d of $a \in \Phi$ from $\Psi \subseteq \Phi$ is a finite tree[1] of height $\mathrm{h}(d)$ labelled by members of Φ such that: (1) a labels the root; (2) for each node labelled a_0, either $a_0 \in \Psi$ or its child nodes are labelled $a_1, \ldots, a_n$ and $a_1, \ldots, a_n \;/\; a_0$ is an instance of a rule of C. If there is a C-derivation d of a from Ψ, then a is said to be C*-derivable* from Ψ, written $d; \Psi \vdash_C a$ or simply $\Psi \vdash_C a$. More generally, a rule is C*-derivable* if for each instance $a_1, \ldots, a_n \;/\; a$ of the rule, a is C-derivable from $\{a_1, \ldots, a_n\}$. Finally, a rule is C*-admissible* if for each instance $a_1, \ldots, a_n \;/\; a$ of the rule, whenever $\vdash_C a_i$ for $i = 1 \ldots n$, then $\vdash_C a$.

Despite the generality of these definitions, in practice we deal exclusively with structures built from formulas. Recall that a propositional language $\mathcal{L}$ consists of a set of symbols (connectives) with fixed arities and that the set of formulas $Fm_{\mathcal{L}}$ is the domain of the absolutely free algebra $\boldsymbol{Fm}_{\mathcal{L}}$ of $\mathcal{L}$ over a fixed countably infinite set of variables. We use the (subscripted) symbols p, q, r and φ, ψ, χ to denote arbitrary variables and formulas, respectively, and define $\mathrm{cp}(\varphi)$, the *complexity* of φ, to be the number of occurrences of connectives in φ. Connectives of arity 0 are called *constants*, while both variables and constants are called *atoms* or *atomic formulas*.

For simplicity, we mostly restrict our attention in what follows to the language $\mathcal{L}_p$ with binary connectives $\wedge, \vee, \&, \to$ and constants $\overline{0}, \overline{1}, \bot, \top$, defining also $\neg\varphi =_{\mathrm{def}} \varphi \to \overline{0}$, $\varphi \oplus \psi =_{\mathrm{def}} \neg\varphi \to \psi$, $\varphi^0 = \overline{1}$, and $\varphi^{n+1} = \varphi \,\&\, \varphi^n$ for $n \in \mathsf{N}$. Note only that in general our methods apply uniformly to fragments of $\mathcal{L}_p$, and in particular to the languages obtained by dropping the constants $\bot, \top$ (e.g., for $\mathrm{FL_e}$-algebras) or $\bot, \top, \overline{0}$ (e.g., for commutative residuated lattices). A useful example of a proof system HMAILL for the set of formulas $Fm_{\mathcal{L}_p}$, corresponding to multiplicative additive intuitionistic linear logic, is presented in Figure 1.

2.2 A proof system for lattices

Let us begin with a simple but relevant case study. Recall that the class $\mathbb{LAT}$ of *lattices* can be defined in at least two ways. Order-theoretically, a lattice is a partially ordered set $\langle L, \leq \rangle$ such that for all $x, y \in L$, both the meet $x \wedge y$ and join $x \vee y$ exist in L.

[1] Recall that a *finite tree* is a finite poset $\langle P, \leq \rangle$ with a distinguished element $x_0 \in P$ called the *root* such that $x_0 \leq x$ for all $x \in P$ and $\langle \{y \in P \mid y \leq x\}, \leq \rangle$ is linearly ordered for all $x \in P$. The members of P are called *nodes* and each node x such that $\{y \in P \mid x < y\} = \emptyset$ is called a *leaf*. For each leaf x, the set $\{y \in P \mid y \leq x\}$ is called a *branch*. A node x is a *child* of a *parent* node y if $y < x$ and $\{z \in P \mid y < z < x\} = \emptyset$. The *height* of the tree is $\sup\{|\{y \in P \mid y \leq x\}| \mid x \in P\}$.

$$
\begin{array}{rl}
\text{(B)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(C)} & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(I)} & \varphi \to \varphi \\
(\&1) & \varphi \to (\psi \to (\varphi \,\&\, \psi)) \\
(\&2) & (\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi) \\
(\overline{1}1) & \overline{1} \\
(\overline{1}2) & \varphi \to (\overline{1} \to \varphi) \\
(\wedge 1) & (\varphi \wedge \psi) \to \varphi \\
(\wedge 2) & (\varphi \wedge \psi) \to \psi \\
(\wedge 3) & ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \\
(\vee 1) & \varphi \to (\varphi \vee \psi) \\
(\vee 2) & \psi \to (\varphi \vee \psi) \\
(\vee 3) & ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi) \\
(\bot) & \bot \to \varphi \\
(\top) & \varphi \to \top
\end{array}
$$

$$
\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\textsc{mp}) \qquad \frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\textsc{adj})
$$

Figure 1. The Hilbert system HMAILL

Algebraically, a lattice is an algebra $\langle L, \wedge, \vee \rangle$ for a language $\mathcal{L}_l$ with binary operations $\wedge$ and $\vee$ such that defining $x \leq y$ iff (if and only if) $x \wedge y = x$ ensures that $\langle L, \leq \rangle$ is a lattice with meet $\wedge$ and join $\vee$ in the order-theoretic sense. Alternatively, lattices are algebras $\langle L, \wedge, \vee \rangle$ satisfying the equations:

$$
\begin{array}{rclrcl}
p \vee (q \vee r) & \approx & (p \vee q) \vee r & \quad p \wedge (q \wedge r) & \approx & (p \wedge q) \wedge r \\
p \wedge q & \approx & q \wedge p & p \vee q & \approx & q \vee p \\
p \wedge (p \vee q) & \approx & p & p \vee (p \wedge q) & \approx & p.
\end{array}
$$

It would be convenient, however, to have also an *algorithmic* presentation of $\mathbb{LAT}$: a method for determining when an arbitrary lattice equation holds in all lattices. Let us therefore restrict our attention to very simple axioms, inequations of the form $\varphi \leq \varphi$ (where $\varphi \leq \psi$ stands for $\varphi \wedge \psi \approx \varphi$), and define rules that decompose formulas by introducing operation symbols on either side of the inequation. We also include (for now) a "cut rule" corresponding to the transitivity of the relation $\leq$. The resulting proof system GLat is displayed in Figure 2, where rules are presented schematically using φ, ψ, χ (subscripted) to stand for arbitrary lattice formulas.

EXAMPLE 2.2.1. *Derivations in* GLat *are finite trees labelled with inequations; e.g.,*

$$
\dfrac{\dfrac{\dfrac{}{q \leq q}\,(\textsc{id})}{p \wedge q \leq q}\,(\wedge\!\Rightarrow)_2 \quad \dfrac{\dfrac{}{p \leq p}\,(\textsc{id})}{p \wedge q \leq p}\,(\wedge\!\Rightarrow)_1}{p \wedge q \leq q \wedge p}\,(\Rightarrow\!\wedge)
\qquad
\dfrac{\dfrac{}{p \leq p}\,(\textsc{id}) \quad \dfrac{}{p \leq p}\,(\textsc{id})}{p \leq p \wedge p}\,(\Rightarrow\!\wedge)
$$

establish commutativity and half of the idempotency law for $\wedge$.

As intended, GLat really is a proof system for $\mathbb{LAT}$. One direction is easy to check. Clearly the axioms hold in all lattices, and it is not hard to see that if the premises of an

Axioms

$$\frac{}{\varphi \leq \varphi}\ (\text{ID})$$

Cut rule

$$\frac{\psi \leq \chi \quad \chi \leq \varphi}{\psi \leq \varphi}\ (\text{CUT})$$

Left logical rules

$$\frac{\varphi_1 \leq \psi}{\varphi_1 \wedge \varphi_2 \leq \psi}\ (\wedge\Rightarrow)_1$$

$$\frac{\varphi_2 \leq \psi}{\varphi_1 \wedge \varphi_2 \leq \psi}\ (\wedge\Rightarrow)_2$$

$$\frac{\varphi_1 \leq \psi \quad \varphi_2 \leq \psi}{\varphi_1 \vee \varphi_2 \leq \psi}\ (\vee\Rightarrow)$$

Right logical rules

$$\frac{\psi \leq \varphi_1}{\psi \leq \varphi_1 \vee \varphi_2}\ (\Rightarrow\vee)_1$$

$$\frac{\psi \leq \varphi_2}{\psi \leq \varphi_1 \vee \varphi_2}\ (\Rightarrow\vee)_2$$

$$\frac{\psi \leq \varphi_1 \quad \psi \leq \varphi_2}{\psi \leq \varphi_1 \wedge \varphi_2}\ (\Rightarrow\wedge)$$

Figure 2. The proof system GLat

instance of a rule hold in all lattices, then so does the conclusion. Hence, by a simple induction on the height of a derivation, $\vdash_{\mathrm{GLat}} \varphi \leq \psi$ implies $\models_{\mathbb{LAT}} \varphi \leq \psi$.[2] For the opposite direction, we define (as for Hilbert systems) a Lindenbaum-Tarski (or free) algebra. First we introduce a binary relation Θ on $Fm_{\mathcal{L}_l}$ by stipulating that

$$(\varphi, \psi) \in \Theta \qquad \text{iff} \qquad (\vdash_{\mathrm{GLat}} \varphi \leq \psi \quad \text{and} \quad \vdash_{\mathrm{GLat}} \psi \leq \varphi).$$

This relation is reflexive by (ID), symmetric by definition, and transitive by (CUT), and therefore an equivalence relation on $Fm_{\mathcal{L}_l}$. In fact, it is a congruence on the lattice formula algebra $\boldsymbol{Fm}_{\mathcal{L}_1}$. For example, if $(\varphi_1, \psi_1) \in \Theta$ and $(\varphi_2, \psi_2) \in \Theta$, then $(\varphi_1 \wedge \varphi_2, \psi_1 \wedge \psi_2) \in \Theta$ as established by the derivations:

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi_1 \leq \psi_1}}{\varphi_1 \wedge \varphi_2 \leq \psi_1}\ (\wedge\Rightarrow)_1 \quad \dfrac{\dfrac{\vdots}{\varphi_2 \leq \psi_2}}{\varphi_1 \wedge \varphi_2 \leq \psi_2}\ (\wedge\Rightarrow)_2}{\varphi_1 \wedge \varphi_2 \leq \psi_1 \wedge \psi_2}\ (\Rightarrow\wedge) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\psi_1 \leq \varphi_1}}{\psi_1 \wedge \psi_2 \leq \varphi_1}\ (\wedge\Rightarrow)_1 \quad \dfrac{\dfrac{\vdots}{\psi_2 \leq \varphi_2}}{\psi_1 \wedge \psi_2 \leq \varphi_2}\ (\wedge\Rightarrow)_2}{\psi_1 \wedge \psi_2 \leq \varphi_1 \wedge \varphi_2}\ (\Rightarrow\wedge)$$

Moreover, the quotient algebra $\boldsymbol{Fm}_{\mathcal{L}_1}/\Theta$ is a lattice. For example, $\wedge$ is commutative in virtue of the derivation:

$$\dfrac{\dfrac{\dfrac{}{\psi \leq \psi}\ (\text{ID})}{\varphi \wedge \psi \leq \psi}\ (\wedge\Rightarrow)_2 \quad \dfrac{\dfrac{}{\varphi \leq \varphi}\ (\text{ID})}{\varphi \wedge \psi \leq \varphi}\ (\wedge\Rightarrow)_1}{\varphi \wedge \psi \leq \psi \wedge \varphi}\ (\Rightarrow\wedge)$$

Observe also that in the algebra $\boldsymbol{Fm}_{\mathcal{L}_1}/\Theta$, for any $\varphi, \psi \in Fm_{\mathcal{L}_l}$:

$$\begin{aligned} \varphi/\Theta \leq \psi/\Theta \quad & \text{iff} \quad \varphi/\Theta \wedge \psi/\Theta = \varphi/\Theta \\ & \text{iff} \quad (\varphi \wedge \psi)/\Theta = \varphi/\Theta \end{aligned}$$

[2] For a class of algebras $\mathbb{K}$ of the same language $\mathcal{L}$ and a set $\Sigma \cup \{\varphi \approx \psi\}$ of equations for $\mathcal{L}$, recall that $\Sigma \models_{\mathbb{K}} \varphi \approx \psi$ means that for each $\boldsymbol{A} \in \mathbb{K}$ and homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$, whenever $\Sigma \subseteq \ker e$, also $\varphi \approx \psi \in \ker e$. By convention, we write $\models_{\mathbb{K}} \varphi \approx \psi$ if $\Sigma = \emptyset$ and $\Sigma \models_{\boldsymbol{A}} \varphi \approx \psi$ if $\mathbb{K} = \{\boldsymbol{A}\}$.

$$\begin{array}{ll} \text{iff} & (\varphi \wedge \psi, \varphi) \in \Theta \\ \text{iff} & \vdash_{\mathrm{GLat}} \varphi \wedge \psi \leq \varphi \ \text{ and } \ \vdash_{\mathrm{GLat}} \varphi \leq \varphi \wedge \psi \\ \text{iff} & \vdash_{\mathrm{GLat}} \varphi \leq \psi. \end{array}$$

Suppose that $\models_{\mathbb{LAT}} \varphi \leq \psi$. Then in particular, $e(\varphi) \leq e(\psi)$ in $\boldsymbol{Fm}_{\mathcal{L}_1}/\Theta$ where e is the natural homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}_1} \to \boldsymbol{Fm}_{\mathcal{L}_1}/\Theta$ defined by $e(\chi) = \chi/\Theta$. So $\varphi/\Theta \leq \psi/\Theta$ in $\boldsymbol{Fm}_{\mathcal{L}_1}/\Theta$, and by the above reasoning, $\vdash_{\mathrm{GLat}} \varphi \leq \psi$. We have therefore established:

THEOREM 2.2.2. $\vdash_{\mathrm{GLat}} \varphi \leq \psi$ *iff* $\models_{\mathbb{LAT}} \varphi \leq \psi$.

Although proof systems can have any axioms and rules whatsoever, they are often designed carefully to have special properties. In particular, notice that every rule of GLat except (CUT) has the so-called "subformula property". Namely, every formula occurring in the premises of an instance of such a rule occurs as a subformula of a formula in the conclusion. Such rules are often also called *analytic* since they decompose (reading upwards) formulas into their constituent subformulas. Derivations involving only these rules are much easier to investigate. In particular, we can see easily that starting with any inequation and applying the rules of GLat except (CUT) backwards, we always terminate with inequations containing only variables.

It would be nice therefore to restrict our attention to "cut-free" derivations in GLat that make no use of the cut rule. This is possible so long as we are concerned only with inequations derivable from the empty set of inequations. For any proof system GL with a cut rule (CUT), let us fix GL° to be GL without (CUT). We will say that GL admits *cut elimination* if there is an algorithm which given a derivation of a structure from $\emptyset$ in GL, outputs a derivation of the structure from $\emptyset$ in GL°.

THEOREM 2.2.3. GLat *admits cut elimination.*

Proof. It is enough to give a constructive proof of the following:

Claim: if $d_1 \vdash_{\mathrm{GLat}^\circ} \varphi \leq \chi$ and $d_2 \vdash_{\mathrm{GLat}^\circ} \chi \leq \psi$, then $\vdash_{\mathrm{GLat}^\circ} \varphi \leq \psi$.

We prove the claim by induction on $\mathrm{h}(d_1) + \mathrm{h}(d_2)$. First, let us consider the case when $\mathrm{h}(d_1) = 0$ or $\mathrm{h}(d_2) = 0$. Suppose without loss of generality that d_1 consists of an instance of the rule (ID). Then $\varphi = \chi$, so clearly $\vdash_{\mathrm{GLat}^\circ} \varphi \leq \psi$ using the derivation d_2. For the inductive step there are two cases.

- First, suppose that one of the derivations d_1 and d_2 ends with an instance of a rule that decomposes φ or ψ, respectively. For example, suppose that $\varphi = \varphi_1 \vee \varphi_2$ and d_1 ends with

$$\dfrac{\dfrac{\vdots}{\varphi_1 \leq \chi} \quad \dfrac{\vdots}{\varphi_2 \leq \chi}}{\varphi_1 \vee \varphi_2 \leq \chi}\ (\vee\Rightarrow)$$

  Let d_{11} and d_{12} be the derivations of $\varphi_1 \leq \chi$ and $\varphi_2 \leq \chi$, respectively. Since $\mathrm{h}(d_{11}) < \mathrm{h}(d_1)$ and $\mathrm{h}(d_{12}) < \mathrm{h}(d_1)$, by two applications of the induction hypothesis, $\vdash_{\mathrm{GLat}^\circ} \varphi_1 \leq \psi$ and $\vdash_{\mathrm{GLat}^\circ} \varphi_2 \leq \psi$. But then by an application of $(\vee\Rightarrow)$, also $\vdash_{\mathrm{GLat}^\circ} \varphi_1 \vee \varphi_2 \leq \psi$ as required.

- Second, suppose that both derivations end with rules that decompose χ. Consider the case where $\chi = \chi_1 \wedge \chi_2$. Then d_1 and d_2 end with

$$\dfrac{\dfrac{\vdots}{\varphi \leq \chi_1} \quad \dfrac{\vdots}{\varphi \leq \chi_2}}{\varphi \leq \chi_1 \wedge \chi_2}\ (\Rightarrow\wedge) \qquad \dfrac{\dfrac{\vdots}{\chi_i \leq \psi}}{\chi_1 \wedge \chi_2 \leq \psi}\ (\wedge\Rightarrow)_i$$

where $i \in \{1, 2\}$. Let d_{1i} be the derivation ending with $\varphi \leq \chi_i$ and let d_{2i} be the derivation ending with $\chi_i \leq \psi$. Since $\mathrm{h}(d_{1i}) < \mathrm{h}(d_1)$ and $\mathrm{h}(d_{2i}) < \mathrm{h}(d_2)$, by the induction hypothesis, $\vdash_{\mathrm{GLat}^\circ} \varphi \leq \psi$, as required. □

Hence, as argued above, we obtain a decision procedure for checking whether an equation holds in all lattices.

COROLLARY 2.2.4. *The equational theory of lattices is decidable.*

Let us also mention that we obtain immediately the admissibility of certain rules for lattices (or, equivalently, validity of universal formulas in free lattices), including *Whitman's condition*:

$$\begin{aligned} \models_{\mathbb{LAT}} \varphi_1 \wedge \varphi_2 \leq \psi_1 \vee \psi_2 \quad \Longrightarrow \quad & \models_{\mathbb{LAT}} \varphi_1 \wedge \varphi_2 \leq \psi_1 \ \text{ or } \ \models_{\mathbb{LAT}} \varphi_1 \wedge \varphi_2 \leq \psi_2 \ \text{ or} \\ & \models_{\mathbb{LAT}} \varphi_1 \leq \psi_1 \vee \psi_2 \ \text{ or } \ \models_{\mathbb{LAT}} \varphi_2 \leq \psi_1 \vee \psi_2. \end{aligned}$$

Just consider the last possible application of a rule in any derivation of an inequation $\varphi_1 \wedge \varphi_2 \leq \psi_1 \vee \psi_2$ in GLat°.

2.3 A sequent calculus for classical logic

Inequations are suitable structures for reasoning about lattices, but most logics and classes of algebras requires something a bit more complicated. Consider, for example, the following inequation corresponding to the distributivity law:

$$(p \vee q) \wedge r \leq (p \wedge r) \vee (q \wedge r).$$

Quite rightly, this inequation is not derivable in GLat. The only step that we could take (working backwards) in a cut-free derivation would be to apply $(\wedge\Rightarrow)_1$, $(\wedge\Rightarrow)_2$, $(\Rightarrow\vee)_1$, or $(\Rightarrow\vee)_2$ and it is easy to see that none of the resulting inequations are derivable for these cases. So what rule(s) might we add to obtain distributive lattices? Somehow we need to represent the $\wedge$ on the left of inequations and perhaps also the $\vee$ on the right.

The solution of Gentzen to this problem was to consider sequences of formulas divided by a sequent arrow, the left sequence representing a conjunction of formulas, the right, a disjunction. However, since we deal only with commutative logics in this chapter, we will make use of multisets rather than sequences of formulas.[3] Finite multisets are typically written using set-like notation with brackets [and] (rather than { and })

[3]Formally, a *multiset over a set* A is an ordered pair $\langle A, f\rangle$ where f is a function $f\colon A \to \mathsf{N}$, called *finite* if $\{x \in A \mid f(x) > 0\}$ is finite. For multisets $\langle A, f_1\rangle$ and $\langle A, f_2\rangle$, $\langle A, f_1\rangle \uplus \langle A, f_2\rangle = \langle A, f\rangle$ with $f(x) = f_1(x) + f_2(x)$; $\langle A, f_1\rangle \subseteq \langle A, f_2\rangle$ iff $f_1(x) \leq f_2(x)$ for all $x \in A$; $x \in \langle A, f\rangle$ iff $x \in A$ and $f(x) > 0$.

where elements can be repeated, e.g., $[a, a, b, b, b]$. Arbitrary finite multisets of formulas will be denoted by (subscripted) upper case Greek letters $\Gamma, \Delta, \Pi, \Sigma$. We often write Γ, Π and Γ, φ to denote the multiset sums $\Gamma \uplus \Pi$ and $\Gamma \uplus [\varphi]$, respectively, φ for the multiset $[\varphi]$, and a blank space for the empty multiset $[]$. Also, for any multiset Γ, we define $\Gamma^0 = []$ and $\Gamma^{n+1} = \Gamma^n \uplus \Gamma$ for all $n \in \mathsf{N}$.

An $\mathcal{L}$-*sequent* S is then an ordered pair of finite multisets of $\mathcal{L}$-formulas, written

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$$

called *single-conclusion* if $m \leq 1$, and *atomic* if $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$ are atoms. Intuitively, we might understand S as "φ_1 and ... and φ_n implies ψ_1 or ... or ψ_m".

A sequent calculus for classical logic (or Boolean algebras) in the language $\mathcal{L}_c$ with binary connectives $\to, \wedge, \vee$ and constants $\top, \bot$ is presented in Figure 3. As in GLat, there are very simple axioms, a cut rule, and rules that introduce connectives on both sides of the sequent arrow. Also present are *structural rules* that do not decompose individual formulas but rather manipulate the structure of the sequents themselves. In particular, the *weakening rules* (WL) and (WR) allow any formula to be added on the left or right of sequents and correspond for lattices to the identities $\varphi \wedge \psi \leq \varphi$ and $\varphi \leq \varphi \vee \psi$, while the *contraction rules* (CL) and (CR) allow two occurrences of a formula to be contracted into one and correspond for lattices to the identities $\varphi \leq \varphi \wedge \varphi$ and $\varphi \leq \varphi \vee \varphi$. These rules suggest reading the "," on the left of sequents as "$\wedge$" and on the right of sequents as "$\vee$"; that is, a sequent $\Gamma \Rightarrow \Delta$ can be interpreted as the formula $\bigwedge \Gamma \to \bigvee \Delta$ where $\bigwedge [] = \top$ and $\bigvee [] = \bot$.

EXAMPLE 2.3.1. *Let us take a look at a derivation in* GCL *of Peirce's law:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID})}{\varphi \Rightarrow \psi, \varphi}\,(\text{WR})}{\Rightarrow \varphi \to \psi, \varphi}\,(\Rightarrow\to) \qquad \dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID})}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi, \varphi}\,(\to\Rightarrow)}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi}\,(\text{CR})}{\Rightarrow ((\varphi \to \psi) \to \varphi) \to \varphi}\,(\Rightarrow\to)$$

EXAMPLE 2.3.2. *The cut rule is closely related to the modus ponens rule used in Hilbert systems. Indeed we can simulate* (MP) *in systems with* (CUT) *as follows:*

$$\dfrac{\Rightarrow \varphi \qquad \dfrac{\Rightarrow \varphi \to \psi \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID}) \quad \dfrac{}{\psi \Rightarrow \psi}\,(\text{ID})}{\varphi, \varphi \to \psi \Rightarrow \psi}\,(\to\Rightarrow)}{\varphi \Rightarrow \psi}\,(\text{CUT})}{\Rightarrow \psi}\,(\text{CUT})$$

Soundness with respect to classical logic – we will write $\models_{\mathrm{CL}} \varphi$ to mean that φ is classically valid – follows easily from the validity of the axioms and the preservation of validity from premises to conclusion of the rules. Completeness for GCL can be established making use of the Lindenbaum-Tarski construction explained above for lattices, and a more involved proof of cut elimination then implies the following result:

THEOREM 2.3.3. $\vdash_{\mathrm{GCL}^\circ} \Gamma \Rightarrow \Delta$ *iff* $\models_{\mathrm{CL}} \bigwedge \Gamma \to \bigvee \Delta$.

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\textsc{id})$$

Cut rule

$$\frac{\Gamma_1, \varphi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})$$

Left logical rules

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot\Rightarrow)$$

$$\frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right logical rules

$$\frac{}{\Gamma \Rightarrow \top, \Delta}\ (\Rightarrow\top)$$

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow\vee)_1$$

$$\frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ (\Rightarrow\wedge)$$

Left structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (\textsc{wl})$$

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (\textsc{cl})$$

Right structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (\textsc{wr})$$

$$\frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (\textsc{cr})$$

Figure 3. The sequent calculus GCL

However, let us sketch here a more direct "semantic" completeness proof for GCL$^\circ$. Consider these alternative conjunction, disjunction, and implication rules:

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)' \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow\vee)' \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)'$$

Each of these rules is easily derivable in GCL$^\circ$ using the corresponding logical rules and the structural rules. E.g., for $(\wedge\Rightarrow)'$ we have the GCL$^\circ$-derivation:

$$\cfrac{\cfrac{\cfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_2}{\Gamma, \varphi \wedge \psi, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_1}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\textsc{cl})$$

Indeed, the converse also holds: namely, the rules $(\wedge\Rightarrow)_1$, $(\wedge\Rightarrow)_2$, $(\Rightarrow\vee)_1$, $(\Rightarrow\vee)_2$, and $(\to\Rightarrow)$ are derivable using $(\wedge\Rightarrow)'$, $(\Rightarrow\vee)'$, and $(\to\Rightarrow)'$ and the structural rules. E.g., for $(\wedge\Rightarrow)_1$ and $(\wedge\Rightarrow)_2$, we have the derivations:

$$\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi, \psi \Rightarrow \Delta}\ (\text{WL})}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)' \qquad \dfrac{\dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi, \psi \Rightarrow \Delta}\ (\text{WL})}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)'$$

Moreover, the rules $(\wedge\Rightarrow)'$, $(\Rightarrow\vee)'$, and $(\rightarrow\Rightarrow)'$, together with the rules $(\bot\Rightarrow)$, $(\Rightarrow\top)$, $(\Rightarrow\rightarrow)$, $(\Rightarrow\wedge)$, and $(\vee\Rightarrow)$ have a useful "invertibility" property. Namely, for any instance

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

of one of these rules:

$$\models_{\text{CL}} \bigwedge \Gamma \rightarrow \bigvee \Delta \qquad \text{iff} \qquad \models_{\text{CL}} \bigwedge \Gamma_i \rightarrow \bigvee \Delta_i \ \text{ for } i = 1 \dots n.$$

This property allows us to prove $\vdash_{\text{GCL}^\circ} \Gamma \Rightarrow \Delta$ iff $\models_{\text{CL}} \bigwedge \Gamma \rightarrow \bigvee \Delta$ by induction on the number of occurrences of connectives in $\Gamma \Rightarrow \Delta$. For the base case, we note that a sequent $\Gamma \Rightarrow \Delta$ containing only propositional variables is classically valid iff there is a common variable on both sides of the sequent: $\Gamma \Rightarrow \Delta$ is then derivable using just weakening rules and (ID). For the inductive step, we choose an occurrence of a connective and apply the corresponding rule and the induction hypothesis. E.g., suppose that $\Gamma = \Gamma' \uplus [\varphi \wedge \psi]$. Then using the aforementioned property of $(\wedge\Rightarrow)'$, together with the induction hypothesis:

$$\begin{aligned} \models_{\text{CL}} \bigwedge \Gamma' \uplus [\varphi \wedge \psi] \rightarrow \bigvee \Delta \quad &\text{iff} \quad \models_{\text{CL}} \bigwedge \Gamma' \uplus [\varphi, \psi] \rightarrow \bigvee \Delta \\ &\text{iff} \quad \vdash_{\text{GCL}^\circ} \Gamma' \uplus [\varphi, \psi] \Rightarrow \Delta. \end{aligned}$$

But $\vdash_{\text{GCL}^\circ} \Gamma' \uplus [\varphi, \psi] \Rightarrow \Delta$ implies $\vdash_{\text{GCL}^\circ} \Gamma' \uplus [\varphi \wedge \psi] \Rightarrow \Delta$ since $(\wedge\Rightarrow)'$ is derivable, and $\vdash_{\text{GCL}^\circ} \Gamma' \uplus [\varphi \wedge \psi] \Rightarrow \Delta$ implies $\models_{\text{CL}} \bigwedge \Gamma' \uplus [\varphi \wedge \psi] \rightarrow \bigvee \Delta$ by soundness, so we have the desired equivalence.

2.4 Substructural logics

A remarkable feature of GCL is that it is transformed into a calculus for intuitionistic logic by applying one simple restriction. Let the set of inferences of a sequent rule (r) with single-conclusion premises and conclusion be called the *single-conclusion version* of (r). The sequent calculus GIL for intuitionistic logic then consists of the single-conclusion versions of the rules of GCL, noting that GIL has no right contraction rules, and right weakening is confined to premises with empty succedents (blocking, for instance, the derivation of Peirce's law in Example 2.3.1). In this sense, intuitionistic logic provides the first example of a "substructural logic": intuitively, a logic that lacks some of the structural rules of classical logic.

Further substructural logics are obtained by removing weakening or contraction rules from GIL and GCL. In such cases, there also arise further natural choices for connectives; in particular (inspired perhaps by the above completeness proof for GCL°) we can introduce rules for an alternative conjunction connective $\&$

$$\dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \,\&\, \psi \Rightarrow \Delta}\ (\&\Rightarrow) \qquad \dfrac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \,\&\, \psi, \Delta_1, \Delta_2}\ (\Rightarrow\&)$$

and for alternative truth constants $\overline{1}$ and $\overline{0}$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \overline{1} \Rightarrow \Delta}\ (\overline{1}\Rightarrow) \qquad \frac{}{\Rightarrow \overline{1}}\ (\Rightarrow\overline{1}) \qquad \frac{}{\overline{0} \Rightarrow}\ (\overline{0}\Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \overline{0}, \Delta}\ (\Rightarrow\overline{0})$$

In GCL and GIL it is possible to prove the sequents $(\varphi \& \psi \Rightarrow \varphi \wedge \psi)$, $(\varphi \wedge \psi \Rightarrow \varphi \& \psi)$, $(\overline{1} \Rightarrow \top)$, $(\top \Rightarrow \overline{1})$, $(\overline{0} \Rightarrow \bot)$, and $(\bot \Rightarrow \overline{0})$, but in logics lacking structural rules, this might no longer be the case. If we add all of these extra rules to GCL without weakening and contraction, we obtain a calculus GMALL, and taking single-conclusion versions, we obtain a calculus GMAILL corresponding to the Hilbert system HMAILL of Figure 1. Removing the rules for $\top$ and $\bot$ gives a calculus for FL_e-algebras, and removing also the rules for $\overline{0}$ gives a calculus for commutative residuated lattices.[4] All these calculi admit cut elimination and have proved extremely useful (indeed essential) in tackling questions such as decidability, complexity, interpolation, etc. for the corresponding logics and classes of algebras.

One of the central themes of this chapter is that proof theory reveals fuzzy logics to be first-class substructural logics. How is this to be understood? Roughly, the expression "substructural" refers to the fact that such logics, which all live in a certain sense "below the surface" of classical logic, fail to admit one or more classically valid structural rules. Most convincingly, logics defined by sequent calculi obtained by removing weakening or contraction rules from GCL or GIL may be deemed substructural, although even in these cases, further logical rules may be added to capture connectives that split in the absence of structural rules. Other structural rules, such as "weaker" versions of weakening or contraction, may also be added, giving a family of logics characterized by cut-free sequent calculi. On the other hand, there are classes of logics (e.g., most relevant logics and, as we will see, most fuzzy logics) that "almost" fit into this framework but require more flexible formalisms than sequents. More perplexing still, there are closely related logics (and classes of algebras) lacking structural rules for which no reasonable cut-free calculus is known. Whether these logics should or should not be considered substructural is perhaps simply a matter of taste.

3 From sequents to hypersequents

We can do a lot with sequents; in particular, we can develop analytic proof systems for a broad range of substructural and other non-classical logics. However, for fuzzy logics, sequents are not quite enough. To see why, consider a potential cut-free derivation of a sequent corresponding to the prelinearity axiom

$$\Rightarrow (p \to q) \vee (q \to p).$$

A cut-free derivation of this in GCL requires both contraction and weakening rules. If one of these is unavailable, then we can either tinker with different sequent calculi – e.g.,

[4]Recall that a *commutative residuated lattice* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{1}\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \&, \overline{1}\rangle$ is a commutative monoid, and $x \,\&\, y \le z$ iff $x \le y \to z$ for all $x, y, z \in A$. *Pointed commutative residuated lattices* or *FL_e-algebras* add an extra nullary operation $\overline{0}$ to the signature, while *bounded FL_e-algebras* add also nullary operations $\bot$ and $\top$ so that $\langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice.

taking non-standard (different to those of GMALL) rules for connectives or interpretations of sequents (this approach is taken for some logics in Section 5) – or extend the sequent framework.

Take another look at $\Rightarrow (p \to q) \lor (q \to p)$. To work further on this sequent, it would be helpful to represent the premises of $(\Rightarrow\lor)_1$ and $(\Rightarrow\lor)_2$ together as

$$\Rightarrow p \to q \quad \text{“or”} \quad \Rightarrow q \to p.$$

We would then be able to operate on $p \to q$ and $q \to p$ in parallel. With this in mind, let us define an $\mathcal{L}$-*hypersequent* to be a finite multiset of $\mathcal{L}$-sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

called *single-conclusion* if each $\Gamma_i \Rightarrow \Delta_i$ for $i \in \{1, \ldots, n\}$ is single-conclusion and *atomic* if each $\Gamma_i \Rightarrow \Delta_i$ for $i \in \{1, \ldots, n\}$ is atomic. We will denote arbitrary hypersequents by the (subscripted) symbols $\mathcal{G}$ and $\mathcal{H}$, writing $\mathcal{G} \mid \mathcal{H}$ and $\mathcal{G} \mid S$ (for a sequent S) to denote the multiset unions $\mathcal{G} \uplus \mathcal{H}$ and $\mathcal{G} \uplus [S]$, respectively.

3.1 The core set of rules

Although we now have new structures to play with, we can still use essentially the same logical rules. Supposing that we are dealing with some fixed language, let the *hypersequent version* of a (single-conclusion) sequent rule (r) be the set of inferences defined by the schema

$$\frac{\mathcal{G} \mid S_1 \ \ldots \ \mathcal{G} \mid S_n}{\mathcal{G} \mid S}$$

where $S_1, \ldots, S_n \ / \ S$ is an instance of (r) and $\mathcal{G}$ is any (single-conclusion) hypersequent. The *single-conclusion version* of a hypersequent rule is the set of inferences of the rule with single-conclusion premises and conclusion.

The core set of hypersequent rules displayed in Figure 4 contains hypersequent versions of the rules of the sequent calculus GMALL described in the previous section together with special structural rules operating at the level of sequents: *external weakening* (EW) and *external contraction* (EC). Essentially these extra rules characterize "|" as an additive disjunction: (EW) adds sequents and (EC) removes them. They provide greater flexibility for recording choices in a derivation. In particular, the rules for $\lor$ on the right and those for $\land$ on the left can be combined as follows:

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \land \psi \Rightarrow \Delta} \ (\land\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \lor \psi, \Delta} \ (\Rightarrow\lor)$$

Let GL be any calculus with (EW) and (EC). Then $(\land\Rightarrow)$ and $(\Rightarrow\lor)$ are derivable in GL extended with $(\land\Rightarrow)_1$, $(\land\Rightarrow)_2$, $(\Rightarrow\lor)_1$, and $(\Rightarrow\lor)_2$, and vice versa: these rules are derivable in GL extended with $(\land\Rightarrow)$ and $(\Rightarrow\lor)$. E.g., to show that $(\Rightarrow\lor)$ is derivable using $(\Rightarrow\lor)_1$ and $(\Rightarrow\lor)_2$, we have:

$$\cfrac{\cfrac{\cfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \varphi \lor \psi, \Delta} \ (\Rightarrow\lor)_2}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \lor \psi, \Delta \mid \Gamma \Rightarrow \varphi \lor \psi, \Delta} \ (\Rightarrow\lor)_1}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \lor \psi, \Delta} \ (\text{EC})$$

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id})$$

Cut rule

$$\frac{\mathcal{G} \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \varphi, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})$$

External weakening

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\textsc{ew})$$

External contraction

$$\frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}\ (\textsc{ec})$$

Left logical rules

$$\frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ (\bot\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \bar{1} \Rightarrow \Delta}\ (\bar{1}\Rightarrow)$$

$$\frac{}{\mathcal{G} \mid \bar{0} \Rightarrow}\ (\bar{0}\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \psi \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \varphi \to \psi \Rightarrow \Delta_1, \Delta_2}\ (\to\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta}\ (\&\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_1$$

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right logical rules

$$\frac{}{\mathcal{G} \mid \Gamma \Rightarrow \top, \Delta}\ (\Rightarrow\top)$$

$$\frac{}{\mathcal{G} \mid \Rightarrow \bar{1}}\ (\Rightarrow\bar{1})$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \bar{0}, \Delta}\ (\Rightarrow\bar{0})$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \psi, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \varphi \,\&\, \psi, \Delta_1, \Delta_2}\ (\Rightarrow\&)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow\vee)_1$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow\vee)_2$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ (\Rightarrow\wedge)$$

Figure 4. The core rule set

On the other hand, it is easily shown that adding (EW) and (EC) to the hypersequent version of a sequent calculus has no effect on which sequents are derivable. To get more, we will need to add rules that allow sequents to interact.

For convenience, let us call a hypersequent calculus a *core rule set extension* if it contains at least the core rules displayed in Figure 4 or their single-conclusion versions. We will consider a great variety of core rule set extensions, but let us first investigate some useful general properties of the logical rules. For a proof system C, a rule is said to be C*-invertible* if for each instance $a_1, \ldots, a_n \,/\, a$ of the rule, whenever a is C-derivable, also a_i is C-derivable for each $i \in \{1, \ldots, n\}$. For example, consider an instance of the rule $(\&\Rightarrow)$ and any core rule set extension GL. If the conclusion $(\mathcal{G} \mid \Gamma, \varphi \mathbin{\&} \psi \Rightarrow \Delta)$ is GL-derivable, then the premise is also GL-derivable as shown below:

$$\dfrac{\dfrac{\dfrac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id}) \quad \dfrac{}{\mathcal{G} \mid \psi \Rightarrow \psi}\ (\textsc{id})}{\mathcal{G} \mid \varphi, \psi \Rightarrow \varphi \mathbin{\&} \psi}\ (\Rightarrow\&) \quad \mathcal{G} \mid \Gamma, \varphi \mathbin{\&} \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}\ (\textsc{cut})$$

So $(\&\Rightarrow)$ is GL-invertible. Indeed, this property holds also for several other logical rules, often either for the left rules or right rules for a connective, but not both. Moreover, if GL contains the core rule set (not just the single-conclusion versions), both the left and right rules for $\neg$ and the right rule for $\oplus$ of the following GL-derivable logical rules are GL-invertible:

$$\dfrac{\mathcal{G} \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \psi \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \varphi \oplus \psi \Rightarrow \Delta_1, \Delta_2}\ (\oplus\Rightarrow) \qquad \dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \oplus \psi, \Delta}\ (\Rightarrow\oplus)$$

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta}\ (\neg\Rightarrow) \qquad \dfrac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \neg\varphi, \Delta}\ (\Rightarrow\neg)$$

LEMMA 3.1.1. *The rules* $(\&\Rightarrow)$, $(\Rightarrow\rightarrow)$, $(\overline{0}\Rightarrow)$, $(\Rightarrow\overline{0})$, $(\overline{1}\Rightarrow)$, $(\Rightarrow\overline{1})$, $(\vee\Rightarrow)$, $(\Rightarrow\wedge)$, $(\bot\Rightarrow)$, $(\Rightarrow\top)$, *and* $(\Rightarrow\neg)$ *are* GL*-invertible for any core rule set extension* GL*, and if* GL *is not restricted to the single-conclusion versions of the core rules, then also* $(\neg\Rightarrow)$ *and* $(\Rightarrow\oplus)$ *are* GL*-invertible.*

Structural rules manipulate sequents and formulas with no regard to their internal composition. If we fix the logical rules of our systems, then it is these manipulations that give each calculus its distinctive properties. Here we consider important examples of two kinds of structural rules: *internal rules* that manipulate formulas within individual sequents, and *external rules* that manipulate whole hypersequents. We also introduce along the way many different calculi for fuzzy logics, collecting these definitions together for the reader's convenience in Table 1.

It is often helpful (in particular, for fitting derivations onto the page) to use rules combined with applications of (EW) and (EC). We will denote such combinations with the superscript *. For example, we might make use of a version of (CUT) where the context side-hypersequents are added rather than merged:

$$\dfrac{\mathcal{G}_1 \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \quad \mathcal{G}_2 \mid \Gamma_2 \Rightarrow \varphi, \Delta_2}{\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{cut})^*$$

The core rules, (CUT) excepted, have the "subformula property": the only formulas occurring in the premises are subformulas of formulas occurring in the conclusion. This ensures that each derivation using only these rules can be viewed (reading upwards) as a decomposition of formulas in which no new material is added. In fact we can decompose to such an extent that (ID) can be restricted to instances containing only variables.

LEMMA 3.1.2. *Let* GL *consist of the logical rules for a sublanguage* $\mathcal{L}$ *of* $\mathcal{L}_p$ *and instances of* (ID) *of the form* $(p \Rightarrow p)$ *where* p *is any variable. Then* $\vdash_{\mathrm{GL}} \varphi \Rightarrow \varphi$ *for all* $\varphi \in Fm_{\mathcal{L}}$.

Proof. We proceed by induction on $\mathrm{cp}(\varphi)$. If φ is a variable p, then $(\varphi \Rightarrow \varphi)$ is a suitable instance of $(\textsc{id})$. If φ is a constant, then the cases of $\bot$ and $\top$ follow immediately using $(\bot\Rightarrow)$ and $(\Rightarrow\top)$, respectively, and if φ is $\overline{0}$ or $\overline{1}$, then we have GL-derivations:

$$\dfrac{\dfrac{}{\Rightarrow \overline{1}}\ (\Rightarrow\overline{1})}{\overline{1} \Rightarrow \overline{1}}\ (\overline{1}\Rightarrow) \qquad\qquad \dfrac{\dfrac{}{\overline{0} \Rightarrow}\ (\overline{0}\Rightarrow)}{\overline{0} \Rightarrow \overline{0}}\ (\Rightarrow\overline{0})$$

Now suppose that φ is $\psi \star \chi$ for $\star \in \{\to, \&, \wedge, \vee\}$. By the induction hypothesis twice, $\vdash_{\mathrm{GL}} \psi \Rightarrow \psi$ and $\vdash_{\mathrm{GL}} \chi \Rightarrow \chi$, and we can construct derivations using the left and right rules for $\star$. E.g., for $\to$ and $\vee$, we have:

$$\dfrac{\dfrac{\psi \Rightarrow \psi \quad \chi \Rightarrow \chi}{\psi \to \chi, \psi \Rightarrow \chi}\ (\to\Rightarrow)}{\psi \to \chi \Rightarrow \psi \to \chi}\ (\Rightarrow\to) \qquad\qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \psi \vee \chi}\ (\Rightarrow\vee)_1 \quad \dfrac{\chi \Rightarrow \chi}{\chi \Rightarrow \psi \vee \chi}\ (\Rightarrow\vee)_2}{\psi \vee \chi \Rightarrow \psi \vee \chi}\ (\vee\Rightarrow)$$

□

3.2 Adding structural rules

Just as extending GMALL or the full Lambek calculus with structural rules defines sequent calculi for other substructural logics, so adding hypersequent versions of these same rules to the core rule set can give calculi for a range of fuzzy (and other) logics. Indeed, now we have a wider choice since we can also add structural rules that manipulate more than one sequent.

Communication

For Hilbert systems, it is the prelinearity and distributivity axioms that are key for characterizing linearity. For Gentzen systems, it is the following communication rule, so-called because formulas are "communicated" between different sequents:

$$\dfrac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\ (\textsc{com})$$

EXAMPLE 3.2.1. *The best way to understand communication is by returning to the tricky prelinearity axioms* $(\varphi \to \psi) \vee (\psi \to \varphi)$*. The following derivation uses* $(\textsc{com})$ *and* $(\textsc{ec})$ *(implicit in the derived rule* $(\Rightarrow\vee)$*) but no other structural rules:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\textsc{id}) \quad \dfrac{}{\psi \Rightarrow \psi}\ (\textsc{id})}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi}\ (\textsc{com})}{\varphi \Rightarrow \psi \mid\Rightarrow \psi \to \varphi}\ (\Rightarrow\to)}{\Rightarrow \varphi \to \psi \mid\Rightarrow \psi \to \varphi}\ (\Rightarrow\to)}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)}\ (\Rightarrow\vee)$$

Notice that the hypersequent $(\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi)$ *two lines down might be read as just a "hypersequent translation" of the formula* $(\varphi \to \psi) \vee (\psi \to \varphi)$*.*

EXAMPLE 3.2.2. *Consider this derivation of a helpful property of disjunction:*

$$\dfrac{\dfrac{}{\varphi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}\ (\textsc{id}) \quad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\textsc{id}) \quad \dfrac{}{\psi \Rightarrow \psi}\ (\textsc{id})}{\psi \Rightarrow \varphi \mid \varphi \Rightarrow \psi}\ (\textsc{com}) \quad \dfrac{}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \psi}\ (\textsc{id})}{\psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}\ (\vee\Rightarrow)}{\varphi \vee \psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}\ (\vee\Rightarrow)$$

We can also use the communication rule to derive distributivity axioms, where the top right hypersequent is derived as above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{(ID)}}{\varphi \wedge (\psi \vee \chi) \Rightarrow \varphi}\,(\wedge\Rightarrow)_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{(ID)}}{\varphi \wedge (\psi \vee \chi) \Rightarrow \varphi}\,(\wedge\Rightarrow)_1 \quad \dfrac{\psi \vee \chi \Rightarrow \psi \mid \psi \vee \chi \Rightarrow \chi}{\psi \vee \chi \Rightarrow \psi \mid \varphi \wedge (\psi \vee \chi) \Rightarrow \chi}\,(\wedge\Rightarrow)_2}{\psi \vee \chi \Rightarrow \psi \mid \varphi \wedge (\psi \vee \chi) \Rightarrow \varphi \wedge \chi}\,(\Rightarrow\wedge)^*}{\varphi \wedge (\psi \vee \chi) \Rightarrow \psi \mid \varphi \wedge (\psi \vee \chi) \Rightarrow \varphi \wedge \chi}\,(\wedge\Rightarrow)_2}{\varphi \wedge (\psi \vee \chi) \Rightarrow \varphi \wedge \psi \mid \varphi \wedge (\psi \vee \chi) \Rightarrow \varphi \wedge \chi}\,(\Rightarrow\wedge)^*}{\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)}\,(\Rightarrow\vee)}{\Rightarrow (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))}\,(\Rightarrow\to)$$

The addition of (COM) gains us a couple of helpful extra invertibilities to add to those in Lemma 3.1.1. Consider the defined rule $(\Rightarrow\vee)$. In any core rule set extension, we can derive the premise of an instance of this rule from its conclusion as follows

$$\dfrac{\dfrac{\varphi \vee \psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi \quad \mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}{\mathcal{G} \mid \varphi \vee \psi \Rightarrow \varphi \mid \Gamma \Rightarrow \psi, \Delta}\,\text{(CUT)}^* \qquad \mathcal{G} \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \psi, \Delta}\,\text{(CUT)}^*$$

where the leftmost hypersequent is derived as in Example 3.2.2.

LEMMA 3.2.3. $(\wedge\Rightarrow)$ *and* $(\Rightarrow\vee)$ *are* GL-*invertible for any core rule set extension* GL *with* (COM).

A close relative of communication, the so-called "split rule", divides one sequent in the premise into two sequents in the conclusion.

$$\dfrac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\,\text{(SPLIT)}$$

EXAMPLE 3.2.4. *Just as the communication rule corresponds to the prelinearity and distributivity axioms, so the split rule corresponds to the law of excluded middle:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,\text{(ID)}}{\Rightarrow \varphi \mid \varphi \Rightarrow}\,\text{(SPLIT)}}{\Rightarrow \varphi \mid \Rightarrow \neg\varphi}\,(\Rightarrow\neg)}{\Rightarrow \varphi \vee \neg\varphi}\,(\Rightarrow\vee)$$

We can now identify Gentzen systems for some elementary fuzzy logics, letting GIUL (for involutive uninorm logic) consist of the core set of rules together with (COM), and GUL (for uninorm logic) consist of the single-conclusion versions of these rules.

Weakening

Weakening axioms (or integrality at the algebraic level) are a common feature of fuzzy logics based on t-norms. In Gentzen systems, weakening rules typically come in pairs, introducing new formulas on the left and right of sequents. Here, for simplicity, we combine these left and right rules into one:

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\,\text{(W)}$$

The system GIMTL (for involutive monoidal t-norm logic) is GIUL extended with $(\textsc{w})$, and GMTL for (monoidal t-norm logic) is GUL extended with the single-conclusion version of $(\textsc{w})$.

EXAMPLE 3.2.5. $(\textsc{w})$ *allows us to prove that* $\overline{0}$ *implies every formula and that* $\overline{1}$ *is implied by every formula:*

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \overline{1}}\;(\textsc{id})}{\varphi \Rightarrow \overline{1}}\;(\textsc{w})}{\Rightarrow \varphi \to \overline{1}}\;(\Rightarrow\to) \qquad \dfrac{\dfrac{\dfrac{}{\overline{0} \Rightarrow}\;(\overline{0}\Rightarrow)}{\overline{0} \Rightarrow \varphi}\;(\textsc{w})}{\Rightarrow \overline{0} \to \varphi}\;(\Rightarrow\to)$$

A weaker version of the weakening rule "mixes" two sequents into one; the empty sequent can also be viewed as a "nullary mix" of no formulas:

$$\dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\;(\textsc{mix}) \qquad \dfrac{}{\mathcal{G} \mid \Rightarrow}\;(\textsc{emp})$$

EXAMPLE 3.2.6. $(\textsc{mix})$ *and* $(\textsc{emp})$ *connect the constants* $\overline{1}$ *and* $\overline{0}$*. The former allows us to prove (not unreasonably) that* $\overline{0}$ *is less true than* $\overline{1}$*; the latter (more bizarrely) allows us to prove the converse:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow}\;(\textsc{emp})}{\overline{1} \Rightarrow}\;(\overline{1}\Rightarrow)}{\overline{1} \Rightarrow \overline{0}}\;(\Rightarrow\overline{0})}{\Rightarrow \overline{1} \to \overline{0}}\;(\Rightarrow\to) \qquad \dfrac{\dfrac{\dfrac{}{\overline{0} \Rightarrow}\;(\overline{0}\Rightarrow) \quad \dfrac{}{\Rightarrow \overline{1}}\;(\Rightarrow\overline{1})}{\overline{0} \Rightarrow \overline{1}}\;(\textsc{mix})}{\Rightarrow \overline{0} \to \overline{1}}\;(\Rightarrow\to)$$

Moreover, $(\textsc{com})$ *is derivable using* $(\textsc{split})$ *and* $(\textsc{mix})$ *(and so is redundant in Gentzen systems with these rules), while* $(\textsc{split})$ *is derivable using* $(\textsc{com})$ *and* $(\textsc{emp})$.

Contraction

The other (with weakening) core structural rules for Gentzen systems are the so-called contraction rules, which reduce the number of occurrences of a formula in a sequent. Typically, we encounter rules that contract a single formula on the left or right. However, as for weakening, we will consider here the more general version:

$$\dfrac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\;(\textsc{c})$$

EXAMPLE 3.2.7. $(\textsc{c})$ *helps us to derive standard contraction axioms as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\;(\textsc{id}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\;(\textsc{id})}{\varphi, \varphi \Rightarrow \varphi \mathbin{\&} \varphi}\;(\Rightarrow\&)}{\varphi \Rightarrow \varphi \mathbin{\&} \varphi}\;(\textsc{c})}{\Rightarrow \varphi \to (\varphi \mathbin{\&} \varphi)}\;(\Rightarrow\to)$$

Adding the single-conclusion version of $(\textsc{c})$ to GMTL gives the hypersequent calculus GG (for Gödel logic). (Or, consider the hypersequent version of a calculus for intuitionistic logic plus $(\textsc{ew})$, $(\textsc{ec})$, and $(\textsc{com})$.) Adding $(\textsc{c})$ to the core rule set plus weakening $(\textsc{w})$ gives a calculus for classical logic.

The situation for calculi without weakening is more complicated. Adding (C) and (MIX) to the core rule set gives a calculus for the non-distributive R-mingle logic $\mathrm{RM}^{\mathrm{ND}}$. A calculus GRM (which does prove distributivity) for R-mingle is defined at the hypersequent level by adding (C) and (MIX) to GIUL. Adding also (EMP) then gives a calculus GIUML (for involutive uninorm mingle logic).

EXAMPLE 3.2.8. *Sometimes structural rules interact in unexpected ways; e.g., in the presence of* (W) *and* (SPLIT)*, we can derive the contraction rule* (C)*:*

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G} \mid \Pi \Rightarrow \Sigma \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\text{SPLIT})}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\text{W})}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\text{EC})$$

Indeed, GMTL *extended with the single-conclusion version of* (SPLIT) *is a single-conclusion hypersequent calculus for classical logic.*

A more complicated rule, which contracts formulas selectively, is the "mingle" rule where (as the name suggests) elements from two sequents are combined into one:

$$\dfrac{\mathcal{G} \mid \Gamma_1, \Pi \Rightarrow \Sigma, \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_1, \Delta_2}\ (\text{MGL})$$

The calculus GUML (for uninorm mingle logic) is GUL extended with the single-conclusion versions of (MGL) and (C).

EXAMPLE 3.2.9. *Mingle axioms are derivable in calculi with* (MGL) *as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID})}{\varphi, \varphi \Rightarrow \varphi}\ (\text{MGL})}{\varphi \,\&\, \varphi \Rightarrow \varphi}\ (\&\Rightarrow)}{\Rightarrow (\varphi \,\&\, \varphi) \to \varphi}\ (\Rightarrow\to)$$

The contraction rule (C) can also be generalized as follows:

$$\dfrac{\mathcal{G} \mid \Gamma, \Pi_1^n \Rightarrow \Sigma_1^n, \Delta \quad \dots \quad \mathcal{G} \mid \Gamma, \Pi_{n-1}^n \Rightarrow \Sigma_{n-1}^n, \Delta}{\mathcal{G} \mid \Gamma, \Pi_1, \dots, \Pi_{n-1} \Rightarrow \Sigma_1, \dots, \Sigma_{n-1}, \Delta}\ (\text{C}_\text{n}) \quad n = 2, 3, \dots$$

Adding (C_n) $(n \geq 2)$ to GIMTL and its single-conclusion version to GMTL gives calculi $\mathrm{GIMTL_n}$ and $\mathrm{GMTL_n}$, respectively, for involutive n-contractive monoidal t-norm logic and n-contractive monoidal t-norm logic.

Other forms of contraction include the contraction of a whole sequent; in particular, adding the following single-conclusion rule to GMTL gives a calculus GSMTL for strict monoidal t-norm logic:

$$\dfrac{\mathcal{G} \mid \Gamma, \Gamma \Rightarrow}{\mathcal{G} \mid \Gamma \Rightarrow}\ (\text{SC}_2)$$

Calculus	Rules	Single-Conclusion
GUL	core rules + (COM)	yes
GIUL	core rules + (COM)	no
GMTL	GUL + (W)	yes
GIMTL	GIUL + (W)	no
$\mathrm{GMTL_n}$	GMTL + ($\mathrm{C_n}$)	yes
$\mathrm{GIMTL_n}$	GIMTL + ($\mathrm{C_n}$)	no
GSMTL	GMTL + ($\mathrm{SC_2}$)	yes
GWNM	GMTL + (WN)	yes
GNM	GIMTL + (N)	no
GG	GMTL + (C)	yes
GUML	GUL + (C) + (MGL)	yes
GRM	GIUL + (C) + (MIX)	no
GIUML	GRM + (EMP)	no

Table 1. Some hypersequent calculi

EXAMPLE 3.2.10. *Using* ($\mathrm{SC_2}$), *we can derive non-contradiction axioms:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi \Rightarrow \varphi}\ (\mathrm{ID})}{\varphi, \neg\varphi \Rightarrow}\ (\neg\Rightarrow)}{\varphi, \varphi \wedge \neg\varphi \Rightarrow}\ (\wedge\Rightarrow)_2}{\varphi \wedge \neg\varphi, \varphi \wedge \neg\varphi \Rightarrow}\ (\wedge\Rightarrow)_1}{\varphi \wedge \neg\varphi \Rightarrow}\ (\mathrm{SC_2})}{\Rightarrow \neg(\varphi \wedge \neg\varphi)}\ (\Rightarrow\neg)$$

Other forms of contraction may get rather complicated. For example, a calculus GWNM for weak nilpotent minimum logic is obtained by adding the following single-conclusion rule to GMTL:

$$\dfrac{\begin{array}{cc}\mathcal{G} \mid \Gamma_1, \Gamma_2, \Pi_2 \Rightarrow \Delta & \mathcal{G} \mid \Pi_1, \Gamma_2, \Pi_2 \Rightarrow \Delta \\ \mathcal{G} \mid \Gamma_1, \Pi_1, \Pi_2 \Rightarrow \Delta & \mathcal{G} \mid \Pi_1, \Pi_1, \Pi_2 \Rightarrow \Delta\end{array}}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \mid \Pi_1, \Pi_2 \Rightarrow \Delta}\ (\mathrm{WN})$$

On the other hand, a calculus for nilpotent minimum logic GNM requires only the addition to GIMTL of the rule:

$$\dfrac{\mathcal{G} \mid \Gamma, \Gamma, \Pi \Rightarrow \Sigma, \Delta, \Delta \quad \mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\mathrm{N})$$

3.3 Soundness and completeness

So far we have introduced many systems (collected in Table 1), but other than their names, there has been little to indicate that these calculi correspond to fuzzy logics. Here we show that this is indeed the case, preferring for reasons of familiarity to connect rules and hypersequent calculi with axioms and Hilbert systems rather than with equations and classes of algebras, noting, however, that the relationship between Hilbert systems and corresponding classes of algebras is established in other chapters of the handbook.

Our first task will therefore be to define a "standard translation" of sequents and hypersequents into formulas:

$$\begin{array}{rcl} \mathrm{I}(\Gamma \Rightarrow \Delta) & =_{\text{def}} & \&\Gamma \to \oplus\Delta \\ \mathrm{I}(S_1 \mid \ldots \mid S_n) & =_{\text{def}} & \mathrm{I}(S_1) \vee \ldots \vee \mathrm{I}(S_n) \end{array}$$

where $\star[\varphi_1, \ldots, \varphi_n] =_{\text{def}} (\varphi_1 \star \ldots \star \varphi_n)$ for $\star \in \{\&, \oplus\}$, $\&[] =_{\text{def}} \overline{1}$, and $\oplus[] =_{\text{def}} \overline{0}$.

EXAMPLE 3.3.1. *Consider a hypersequent*

$$\mathcal{G} = (\varphi, \varphi \to \chi \Rightarrow \psi \mid\Rightarrow \varphi, \psi \mid \varphi, \psi, \chi \Rightarrow).$$

To find the standard interpretation of $\mathcal{G}$, we first interpret the sequents

$$\begin{array}{rcl} \mathrm{I}(\varphi, \varphi \to \chi \Rightarrow \psi) & = & (\varphi \,\&\, (\varphi \to \chi)) \to \psi \\ \mathrm{I}(\Rightarrow \varphi, \psi) & = & \overline{1} \to (\varphi \oplus \psi) \\ \mathrm{I}(\varphi, \psi, \chi \Rightarrow) & = & (\varphi \,\&\, \psi \,\&\, \chi) \to \overline{0} \end{array}$$

and then take the disjunction, to get

$$\mathrm{I}(\mathcal{G}) = ((\varphi \,\&\, (\varphi \to \chi)) \to \psi) \vee (\overline{1} \to (\varphi \oplus \psi)) \vee ((\varphi \,\&\, \psi \,\&\, \chi) \to \overline{0}).$$

Showing that a Gentzen system GL is sound and complete with respect to a Hilbert system HL consists of showing that $\vdash_{\mathrm{GL}} \mathcal{G}$ iff $\vdash_{\mathrm{HL}} \mathrm{I}(\mathcal{G})$. Both directions involve relatively straightforward inductions on the height of derivations. Let us take as our starting point the hypersequent calculus GUL and the Hilbert system HUL obtained by extending HMAILL (see Figure 1) with the prelinearity and distributivity axioms

$$\begin{array}{ll} \text{(PRL)} & (\varphi \to \psi) \vee (\psi \to \varphi) \\ \text{(DIS)} & (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)). \end{array}$$

To show (inductively) that $\vdash_{\mathrm{GUL}} \mathcal{G}$ implies $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G})$, we have the rather onerous task of establishing for each instance $\mathcal{G}_1 \ldots \mathcal{G}_n / \mathcal{G}$ of a rule of GUL, that $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G}_i)$ for $i = 1 \ldots n$ implies $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G})$. Let us just consider one of the harder cases: a single-conclusion instance of the communication rule. Suppose that

$$\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1) \quad \text{and} \quad \vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2).$$

We want to show

$$\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G} \mid \Gamma_1, \Pi_2 \Rightarrow \Delta_1 \mid \Gamma_2, \Pi_1 \Rightarrow \Delta_2).$$

For convenience, let us define

$$\varphi_1 = \mathrm{I}(\Gamma_1 \Rightarrow \Delta_1), \quad \varphi_2 = \mathrm{I}(\Gamma_2 \Rightarrow \Delta_2), \quad \psi_1 = \&\Pi_1, \quad \psi_2 = \&\Pi_2, \quad \chi = \mathrm{I}(\mathcal{G}).$$

Suppose then that

$$\vdash_{\mathrm{HUL}} (\psi_1 \to \varphi_1) \vee \chi \quad \text{and} \quad \vdash_{\mathrm{HUL}} (\psi_2 \to \varphi_2) \vee \chi.$$

Making use of the following instances of the (B) axiom schema

$$(\psi_1 \to \varphi_1) \to ((\varphi_1 \to \varphi_2) \to (\psi_1 \to \varphi_2))$$
$$\text{and} \quad (\psi_2 \to \varphi_2) \to ((\varphi_2 \to \varphi_1) \to (\psi_2 \to \varphi_1))$$

we obtain (using also various HMAILL-derivable formulas)

$$\vdash_{\mathrm{HUL}} ((\varphi_1 \to \varphi_2) \to (\psi_1 \to \varphi_2)) \vee \chi \quad \text{and} \quad \vdash_{\mathrm{HUL}} ((\varphi_2 \to \varphi_1) \to (\psi_2 \to \varphi_1)) \vee \chi.$$

So using the adjunction rule (ADJ) and the distributivity axiom schema (DIS)

$$\vdash_{\mathrm{HUL}} (((\varphi_1 \to \varphi_2) \to (\psi_1 \to \varphi_2)) \wedge ((\varphi_2 \to \varphi_1) \to (\psi_2 \to \varphi_1))) \vee \chi.$$

Making use of the axioms of HMAILL for $\wedge$ and $\vee$, we obtain

$$\vdash_{\mathrm{HUL}} (((\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_1)) \to ((\psi_1 \to \varphi_2) \vee (\psi_2 \to \varphi_1))) \vee \chi$$

and, since $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_1)$ is an instance of (PRL), finally

$$\vdash_{\mathrm{HUL}} ((\psi_2 \to \varphi_1) \vee (\psi_1 \to \varphi_2)) \vee \chi.$$

To show that $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G})$ implies $\vdash_{\mathrm{GUL}} \mathcal{G}$, we observe first that for each axiom φ of HUL, the sequent $(\Rightarrow \varphi)$ is GUL-derivable. In particular, the cases of (PRL) and (DIS) are established in Examples 3.2.1 and 3.2.2, respectively. Moreover, GUL-derivability is preserved by (MP) (see Example 2.3.2) and (ADJ) (using the $(\Rightarrow\wedge)$ rule). Hence an induction on the height of a derivation in HUL establishes the intermediate result that $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G})$ implies $\vdash_{\mathrm{GUL}} \Rightarrow \mathrm{I}(\mathcal{G})$. Suppose then that

$$\vdash_{\mathrm{GUL}} \Rightarrow (\&\Gamma_1 \to \oplus\Delta_1) \vee \ldots \vee (\&\Gamma_n \to \oplus\Delta_n).$$

By the GUL-invertibility of $(\Rightarrow\vee)$ and $(\Rightarrow\to)$ (Lemmas 3.1.1 and 3.2.3), we obtain

$$\vdash_{\mathrm{GUL}} \&\Gamma_1 \Rightarrow \oplus\Delta_1 \mid \ldots \mid \&\Gamma_n \Rightarrow \oplus\Delta_n.$$

But then using the GUL-invertibility of $(\&\Rightarrow)$, $(\overline{1}\Rightarrow)$, and $(\Rightarrow\overline{0})$ (Lemma 3.1.1)

$$\vdash_{\mathrm{GUL}} \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

So $\vdash_{\mathrm{HUL}} \mathrm{I}(\mathcal{G})$ implies $\vdash_{\mathrm{GUL}} \mathcal{G}$ as required.

A very similar proof establishes that the calculus GIUL (the core rule set plus (COM)) is sound and complete with respect to the Hilbert system HIUL obtained by adding to HUL the involution axioms

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi.$$

Moreover, from an analysis of the above reasoning, we obtain sufficient conditions for an extension of GUL or GIUL to be sound and complete with respect to an extension of HUL or HIUL. Let us say that a hypersequent rule (r) and axiom schema $\mathcal{A}$ are L-*matching* if

1. For every instance $\mathcal{G}_1, \ldots, \mathcal{G}_n \ / \ \mathcal{G}$ of (r):

$$\vdash_{\mathrm{HL}+\mathcal{A}} \mathrm{I}(\mathcal{G}_i) \ \text{ for } \ i = 1 \ldots n \quad \text{implies} \quad \vdash_{\mathrm{HL}+\mathcal{A}} \mathrm{I}(\mathcal{G}).$$

2. $\vdash_{\mathrm{GL}+(r)} \Rightarrow \varphi$ for every axiom φ of $\mathcal{A}$.

Examples of matching rules and axiom schema are displayed in Table 2.

Structural Rule	Matching Axiom Schema
$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}$ (W)	$(\varphi \to \overline{1}) \wedge (\overline{0} \to \varphi)$
$\dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$ (MIX)	$\overline{0} \to \overline{1}$
$\dfrac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$ (SPLIT)	$\varphi \vee \neg\varphi$
$\dfrac{}{\mathcal{G} \mid \Rightarrow}$ (EMP)	$\overline{1} \to \overline{0}$
$\dfrac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}$ (C)	$\varphi \to (\varphi \,\&\, \varphi)$
$\dfrac{\mathcal{G} \mid \Gamma_1, \Pi \Rightarrow \Sigma, \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2, \Pi \Rightarrow \Sigma, \Delta_1, \Delta_2}$ (MGL)	$(\varphi \,\&\, \varphi) \to \varphi$
$\dfrac{\mathcal{G} \mid \Gamma, \Pi_1^n \Rightarrow \Sigma_1^n, \Delta \quad \ldots \quad \mathcal{G} \mid \Gamma, \Pi_{n-1}^n \Rightarrow \Sigma_{n-1}^n, \Delta}{\mathcal{G} \mid \Gamma, \Pi_1, \ldots, \Pi_{n-1} \Rightarrow \Sigma_1, \ldots, \Sigma_{n-1}, \Delta}$ ($\mathrm{C_n}$)	$\varphi^{n-1} \to \varphi^n$
$\dfrac{\begin{array}{ll}\mathcal{G} \mid \Gamma_1, \Gamma_2, \Pi_2 \Rightarrow \Delta & \mathcal{G} \mid \Pi_1, \Gamma_2, \Pi_2 \Rightarrow \Delta \\ \mathcal{G} \mid \Gamma_1, \Pi_1, \Pi_2 \Rightarrow \Delta & \mathcal{G} \mid \Pi_1, \Pi_1, \Pi_2 \Rightarrow \Delta\end{array}}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \mid \Pi_1, \Pi_2 \Rightarrow \Delta}$ (WN)	$\neg(\varphi \,\&\, \psi) \vee ((\varphi \wedge \psi) \to (\varphi \,\&\, \psi))$
$\dfrac{\mathcal{G} \mid \Gamma, \Gamma, \Pi \Rightarrow \Sigma, \Delta, \Delta \quad \mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}$ (N)	$((\varphi^2 \to \psi) \wedge ((\neg\psi)^2 \to \neg\varphi)) \to (\varphi \to \psi)$
$\dfrac{\mathcal{G} \mid \Gamma, \Gamma \Rightarrow \Delta, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$ ($\mathrm{SC_2}$)	$\neg(\varphi \wedge \neg\varphi)$

Table 2. Matching rules and axioms

THEOREM 3.3.2. *Let* L *be* UL *or* IUL *and suppose that* (r_i) *and* $\mathcal{A}_i$ *are* L*-matching for* $i = 1 \ldots n$. *Then*

$$\vdash_{\mathrm{GL}+(\mathrm{r}_1)+\ldots+(\mathrm{r}_n)} \mathcal{G} \quad \text{iff} \quad \vdash_{\mathrm{HL}+\mathcal{A}_1+\ldots+\mathcal{A}_n} \mathrm{I}(\mathcal{G}).$$

4 Eliminations and applications

The hypersequent calculi defined in the previous section provide a uniform and natural presentation of a wide range of fuzzy logics. But are they useful? Proof search in Hilbert systems is hindered by the need to guess formulas φ and $\varphi \to \psi$ as premises when applying modus ponens. However, the same situation seems to occur for Gentzen systems: we have to guess which formula φ to use when applying (CUT). If we could do without the cut rule, then we could just apply rules where formulas in the premises are subformulas of formulas in the conclusion. This "subformula property" (or analyticity of the cut-free calculus) can be useful for, among other things, establishing decidability, complexity, interpolation, and conservative extension results. Here we show that for calculi satisfying certain properties we can algorithmically transform derivations in the calculus with (CUT) into derivations without this rule; that is, we "eliminate" (CUT) from derivations. We also provide similar elimination results for the so-called "density rule". Since the hypersequent calculi extended with the density rule are complete with respect to dense linearly ordered algebras, density elimination implies the same completeness result for the original calculi and provides the key step for standard completeness proofs for the corresponding fuzzy logics.

4.1 Cut elimination

Let us start with an example. Suppose that we want to eliminate an application of (CUT) from an otherwise cut-free derivation in the single-conclusion calculus GMAILL:

$$\frac{\dfrac{\vdots}{\Gamma, \varphi \Rightarrow \Delta} \quad \dfrac{\vdots}{\Pi \Rightarrow \varphi}}{\Gamma, \Pi \Rightarrow \Delta}\ \text{(CUT)}$$

The "cut formula" φ occurs on the left in one premise, and on the right in the other. A natural strategy for eliminating this application of (CUT) is to look more carefully at the derivations of these premises. If one of the premises is an instance of (ID), then it must be $(\varphi \Rightarrow \varphi)$ and the other premise must be $(\Gamma, \Pi \Rightarrow \Delta)$. Otherwise, we have two possibilities. The first is that one of the premises ends with an application of a rule where φ is not decomposed, e.g., letting $\Gamma = \Gamma_1 \uplus \Gamma_2 \uplus [\psi \to \chi]$:

$$\frac{\dfrac{\dfrac{\vdots}{\Gamma_1, \chi, \varphi \Rightarrow \Delta} \quad \dfrac{\vdots}{\Gamma_2 \Rightarrow \psi}}{\Gamma_1, \Gamma_2, \psi \to \chi, \varphi \Rightarrow \Delta}\ (\to\Rightarrow) \quad \dfrac{\vdots}{\Pi \Rightarrow \varphi}}{\Gamma_1, \Gamma_2, \psi \to \chi, \Pi \Rightarrow \Delta}\ \text{(CUT)}$$

In this case, we can "push the cut upwards" in the derivation to get:

$$\frac{\dfrac{\dfrac{\vdots}{\Gamma_1, \chi, \varphi \Rightarrow \Delta} \quad \dfrac{\vdots}{\Pi \Rightarrow \varphi}}{\Gamma_1, \chi, \Pi \Rightarrow \Delta}\ \text{(CUT)} \quad \dfrac{\vdots}{\Gamma_2 \Rightarrow \psi}}{\Gamma_1, \Gamma_2, \psi \to \chi, \Pi \Rightarrow \Delta}\ (\to\Rightarrow)$$

That is, we have a derivation where the left premise in the new application of (CUT) has a shorter derivation than the application in the original derivation.

The second possibility is that φ is decomposed in the last application of a rule in both premises, e.g., with $\Gamma = \Gamma_1 \uplus \Gamma_2$ and $\varphi = \psi \to \chi$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \Rightarrow \psi} \quad \dfrac{\vdots}{\Gamma_2, \chi \Rightarrow \Delta}}{\Gamma_1, \Gamma_2, \psi \to \chi \Rightarrow \Delta}\,(\to\Rightarrow) \quad \dfrac{\dfrac{\vdots}{\Pi, \psi \Rightarrow \chi}}{\Pi \Rightarrow \psi \to \chi}\,(\Rightarrow\to)}{\Gamma_1, \Gamma_2, \Pi \Rightarrow \Delta}\,(\text{CUT})$$

In this case we rearrange our derivation in a different way: we replace the application of (CUT) with applications of (CUT) with cut formulas ψ and χ:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_2, \chi \Rightarrow \Delta} \quad \dfrac{\vdots}{\Pi, \psi \Rightarrow \chi}}{\Gamma_2, \Pi, \psi \Rightarrow \Delta}\,(\text{CUT}) \quad \dfrac{\vdots}{\Gamma_1 \Rightarrow \psi}}{\Gamma_1, \Gamma_2, \Pi \Rightarrow \Delta}\,(\text{CUT})$$

We now have two applications of (CUT) but with cut formulas of a smaller complexity than the original application.

This procedure, formalized using a double induction on cut formula complexity and the combined height of derivations of the premises, eliminates applications of (CUT) for many sequent calculi. However, it encounters a problem with rules that contract formulas in one or more of the premises. Consider the following situation:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \varphi, \varphi \Rightarrow \Delta}}{\Gamma, \varphi \Rightarrow \Delta}\,(\text{C}) \quad \dfrac{\vdots}{\Pi \Rightarrow \varphi}}{\Gamma, \Pi \Rightarrow \Delta}\,(\text{CUT})$$

In this case we need to perform several cuts at once, using a rule something like:

$$\dfrac{\Gamma, [\varphi]^n \Rightarrow \Delta \quad \Pi \Rightarrow \varphi}{\Gamma, \Pi^n \Rightarrow \Delta}$$

For hypersequent calculi, the situation is further complicated by the fact that whole sequents may be contracted using (EC). This means that a cut formula occurring in the premises of an application of (CUT) may appear in several sequents in a hypersequent higher up in the derivation, e.g.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \varphi \Rightarrow \Delta \mid \Gamma, \varphi \Rightarrow \Delta}}{\Gamma, \varphi \Rightarrow \Delta}\,(\text{EC}) \quad \dfrac{\vdots}{\Pi \Rightarrow \varphi}}{\Gamma, \Pi \Rightarrow \Delta}\,(\text{CUT})$$

To cope with this situation, we should eliminate even more general versions of (CUT) that perform multiple cuts in different sequents. This method of cut elimination applies to a broad class of hypersequent calculi satisfying a "substitutivity" condition that allows cuts to be pushed upwards in derivations.

To explain the general method, we make use of some definitions and notational conveniences. First, let us assume that $\lambda, \mu, m, n, i, j, k$ always denote natural numbers and $\Gamma, \Delta, \Pi, \Sigma$ multisets of formulas, recalling that $\Gamma^0 = []$ and $\Gamma^{n+1} = \Gamma \uplus \Gamma^n$. S denotes a sequent and $\mathcal{G}, \mathcal{H}$ hypersequents, and we let $[\mathcal{G}_i]_{i=1}^n$ denote the hypersequent $\mathcal{G}_1 \mid \ldots \mid \mathcal{G}_n$ and $\{\mathcal{G}_i\}_{i=1}^n$ denote the set of hypersequents $\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$.

A *marked hypersequent* is a hypersequent with exactly one occurrence of a formula φ distinguished, written $(\mathcal{G} \mid \Gamma, \underline{\varphi} \Rightarrow \Delta)$ or $(\mathcal{G} \mid \Gamma \Rightarrow \underline{\varphi}, \Delta)$. For a hypersequent $\mathcal{G}$ and a marked hypersequent $\mathcal{H}$, we define the set $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ of results of applying (CUT) multiple times as follows:

(1) If φ does not occur in $\biguplus_{i=1}^n \Gamma_i$ where

$$\mathcal{G} = [\Gamma_i, [\varphi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \qquad \text{and} \qquad \mathcal{H} = (\mathcal{H}' \mid \Pi \Rightarrow \underline{\varphi}, \Sigma),$$

then the set $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ contains, for $0 \leq \mu_i \leq \lambda_i$ $(i = 1 \ldots n)$,

$$\mathcal{H}' \mid [\Gamma_i, \Pi^{\mu_i}, [\varphi]^{\lambda_i - \mu_i} \Rightarrow \Sigma^{\mu_i}, \Delta_i]_{i=1}^n.$$

(2) If φ does not occur in $\biguplus_{i=1}^n \Delta_i$ where

$$\mathcal{G} = [\Gamma_i \Rightarrow [\varphi]^{\lambda_i}, \Delta_i]_{i=1}^n \qquad \text{and} \qquad \mathcal{H} = (\mathcal{H}' \mid \Pi, \underline{\varphi} \Rightarrow \Sigma),$$

then the set $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ contains for $0 \leq \mu_i \leq \lambda_i$ $(i = 1 \ldots n)$,

$$\mathcal{H}' \mid [\Gamma_i, \Pi^{\mu_i} \Rightarrow [\varphi]^{\lambda_i - \mu_i}, \Sigma^{\mu_i}, \Delta_i]_{i=1}^n.$$

One of the crucial steps for our cut elimination method will be shifting (multiple) applications of (CUT) upwards over applications of other rules. We define a (structural) rule (r) to be *substitutive* if for any

1. instance $\mathcal{G}_1, \ldots, \mathcal{G}_n \;/\; \mathcal{G}$ of (r)
2. marked hypersequent $\mathcal{H}$ (single-conclusion if (r) is single-conclusion)
3. $\mathcal{G}' \in \mathrm{CUT}(\mathcal{G}, \mathcal{H})$

there exist $\mathcal{G}'_i \in \mathrm{CUT}(\mathcal{G}_i, \mathcal{H})$ for $i = 1 \ldots n$ such that

$$\frac{\mathcal{G}'_1 \quad \ldots \quad \mathcal{G}'_n}{\mathcal{G}'} \quad \text{is an instance of } (r).$$

The name "substitutive" is apt because the condition implies that substituting occurrences of φ with Π on the left and Σ on the right, in both the conclusion of a rule instance and suitably in its premises, gives another instance of the rule. In particular, it is easy to see that the structural rules introduced in Table 2 are substitutive, since (1) each multiset variable $\Gamma, \Pi, \Delta, \Sigma, \ldots$ and hypersequent variable $\mathcal{G}$ occurs at most once in the conclusion, (2) every multiset or hypersequent variable occurring in a premise occurs in the conclusion, and (3) multiset variables are paired so that, e.g., in the single-conclusion case, any multiset variable occurring on the right in the conclusion occurs always with another multiset variable on the left. However, for example, the following "anti-contraction" rule is not substitutive:

$$\frac{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta}{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}$$

Just consider an instance of the form $(p \Rightarrow \;\; / \; p, p \Rightarrow \;)$ and a marked hypersequent $(q \Rightarrow \underline{p})$. Neither $(p \Rightarrow \;\; / \; p, q \Rightarrow \;)$ nor $(q \Rightarrow \;\; / \; p, q \Rightarrow \;)$ is an instance of the rule.

Note, moreover, that each core logical rule (r) is not substitutive but rather *almost substitutive* in the sense that the rule obtained by removing the principal formula from the conclusion of instances and its decomposed parts from their premises, is substitutive.

Before embarking on our general cut elimination proof, let us review some of the main ideas. First, note that it is enough just to consider an uppermost application of (CUT) in a derivation. If we can remove such an application without introducing any new applications, then we can eliminate applications of (CUT) one by one. So, recalling that GL° is the calculus GL without (CUT), to establish cut elimination for GL it is enough to show constructively that if the premises of an instance of (CUT) are GL°-derivable, then the conclusion is GL°-derivable. Following our earlier discussion, we show more generally that if two hypersequents $\mathcal{G}$ and $\mathcal{H}$ are derivable, the first with a marked occurrence of φ, then any hypersequent in $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ is derivable.

THEOREM 4.1.1. *Cut elimination holds for any extension of* GUL *with substitutive single-conclusion rules or extension of* GIUL *with substitutive rules.*

Proof. Let GL be an extension of GUL with substitutive single-conclusion rules or an extension of GIUL with substitutive rules. As observed above, it suffices to show that an "uppermost" application of (CUT) in any GL-derivation (i.e., where the premises are GL°-derivable) can be eliminated without introducing new applications of (CUT). Hence it is enough to prove that for any hypersequent $\mathcal{G}$ and marked hypersequent $\mathcal{H}$ with marked formula φ:

$$\text{If } \; d_{\mathcal{G}} \vdash_{\mathrm{GL}^\circ} \mathcal{G} \; \text{ and } \; d_{\mathcal{H}} \vdash_{\mathrm{GL}^\circ} \mathcal{H}, \; \text{ then } \vdash_{\mathrm{GL}^\circ} \mathcal{G}' \text{ for all } \mathcal{G}' \in \mathrm{CUT}(\mathcal{G}, \mathcal{H}).$$

We prove the claim by a triple induction on the lexicographically ordered triple

$$\langle \mathrm{cp}(\varphi), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d_{\mathcal{G}}) \rangle$$

where $\mathrm{e}(d) = \begin{cases} 0 & \text{if } d \text{ ends with a logical rule applied to a marked principal formula} \\ 1 & \text{otherwise.} \end{cases}$

We begin by considering the last application of a rule (r) in $d_{\mathcal{G}}$. If (r) is (ID), then $\mathcal{G}$ is of the form $(\mathcal{G}' \mid \chi \Rightarrow \chi)$. So every member of $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ is of the form $(\mathcal{H}' \mid \mathcal{G})$ or $(\mathcal{H}' \mid \mathcal{H})$, and the claim follows using (EW). Otherwise, there are two cases:

(a) The application of (r) is of the form:

$$\frac{\mathcal{G}_1 \quad \ldots \quad \mathcal{G}_n}{\mathcal{G}}$$

and the principal formula (if there is one) is *not* an occurrence of φ on the opposite side to the marked occurrence in $\mathcal{H}$. Pick $\mathcal{G}' \in \mathrm{CUT}(\mathcal{G}, \mathcal{H})$. By the substitutivity of (r) or almost-substitutivity if (r) is a logical rule, there exist $\mathcal{G}'_i \in \mathrm{CUT}(\mathcal{G}_i, \mathcal{H})$ for $i = 1 \ldots n$ such that

$$\frac{\mathcal{G}'_1 \quad \ldots \quad \mathcal{G}'_n}{\mathcal{G}'} \quad \text{is an instance of } (r).$$

But, by the induction hypothesis, $\vdash_{\mathrm{GL}^\circ} \mathcal{G}'_i$ for $i = 1 \ldots n$, so also $\vdash_{\mathrm{GL}^\circ} \mathcal{G}'$ as required.

(b) The application of (r) is of the form:

$$\frac{\mathcal{G}_1 \quad \ldots \quad \mathcal{G}_n}{\mathcal{G}' \mid \Gamma, [\varphi]^\lambda \Rightarrow \Delta} \quad \text{or} \quad \frac{\mathcal{G}_1 \quad \ldots \quad \mathcal{G}_n}{\mathcal{G}' \mid \Gamma \Rightarrow [\varphi]^\lambda, \Delta}$$

where the principal formula is an occurrence of φ on the opposite side to the marked occurrence in $\mathcal{H}$, and $\varphi \notin \Gamma$ or $\varphi \notin \Delta$, respectively. Pick $\mathcal{G}^{\mathcal{H}} \in \mathrm{CUT}(\mathcal{G}, \mathcal{H})$ where $\mathcal{H}$ is of the form, respectively,

$$\mathcal{H}' \mid \Pi \Rightarrow \underline{\varphi}, \Sigma \qquad \text{or} \qquad \mathcal{H}' \mid \Pi, \underline{\varphi} \Rightarrow \Sigma.$$

The only tricky case (others follow as in case (a) using almost-substitutivity) is when $\mathcal{G}^{\mathcal{H}}$ is of the form

$$\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^\lambda \Rightarrow \Sigma^\lambda, \Delta$$

with $\mathcal{G}'' \in \mathrm{CUT}(\mathcal{G}', \mathcal{H})$. Then we also have a member of $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ of the form, respectively,

$$\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1}, \varphi \Rightarrow \Sigma^{\lambda-1}, \Delta \qquad \text{or} \qquad \mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1} \Rightarrow \varphi, \Sigma^{\lambda-1}, \Delta.$$

Since (r) is almost-substitutive, there exist $\mathcal{G}'_i \in \mathrm{CUT}(\mathcal{G}_i, \mathcal{H})$ for $i = 1 \ldots n$ so that we have an instance of (r) of the form, respectively,

$$\frac{\mathcal{G}'_1 \quad \ldots \quad \mathcal{G}'_n}{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1}, \varphi \Rightarrow \Sigma^{\lambda-1}, \Delta} \quad \text{or} \quad \frac{\mathcal{G}'_1 \quad \ldots \quad \mathcal{G}'_n}{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1} \Rightarrow \varphi, \Sigma^{\lambda-1}, \Delta}$$

Moreover, by the induction hypothesis, $\vdash_{\mathrm{GL}^\circ} \mathcal{G}'_i$ for $i = 1 \ldots n$, so we have a derivation d ending with such an application of (r).

Now we consider two subcases:

1. $\mathrm{e}(d_{\mathcal{H}}) = 1$: i.e., $d_{\mathcal{H}}$ does not end with the application of a logical rule to the marked occurrence of φ. Mark the remaining occurrence of φ on the left or right as appropriate in d and remove the marking of φ in $d_{\mathcal{H}}$. So $\mathrm{e}(d) = 0$ and

$$\langle \mathrm{cp}(\varphi), \mathrm{e}(d), \mathrm{h}(d_{\mathcal{H}}) \rangle < \langle \mathrm{cp}(\varphi), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d_{\mathcal{G}}) \rangle.$$

Hence by the induction hypothesis and a further application of (EC), $\vdash_{\mathrm{GL}^\circ} \mathcal{G}^{\mathcal{H}}$.

2. $\mathrm{e}(d_{\mathcal{H}}) = 0$: i.e., $d_{\mathcal{H}}$ ends with the application of a logical rule to the marked occurrence of φ, and is of the form, respectively,

$$\frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_m}{\mathcal{H}' \mid \Pi \Rightarrow \underline{\varphi}, \Sigma} \qquad \text{or} \qquad \frac{\mathcal{H}_1 \quad \ldots \quad \mathcal{H}_m}{\mathcal{H}' \mid \Pi, \underline{\varphi} \Rightarrow \Sigma}$$

Then $\mathcal{G}^{\mathcal{H}} = (\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda} \Rightarrow \Sigma^{\lambda}, \Delta)$ is derivable from $\mathcal{G}'_1, \ldots, \mathcal{G}'_n$, and $\mathcal{H}_1, \ldots, \mathcal{H}_m$ by cuts on subformulas $\varphi_1, \ldots, \varphi_k$ of φ where

$$\langle \mathrm{cp}(\varphi_i), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d) \rangle < \langle \mathrm{cp}(\varphi), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d_{\mathcal{G}}) \rangle \quad \text{for } i = 1 \ldots k.$$

It follows, using (EW) and the induction hypothesis (several times perhaps), that $\vdash_{\mathrm{GL}^{\circ}} \mathcal{G}^{\mathcal{H}}$. For example, suppose that GL is a single-conclusion calculus, $\varphi = \psi \to \chi$, and d ends with

$$\frac{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma_1, \Pi^{\mu} \Rightarrow \psi \quad \mathcal{H}' \mid \mathcal{G}'' \mid \Gamma_2, \Pi^{\lambda-1-\mu}, \chi \Rightarrow \Delta}{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma_1, \Gamma_2, \Pi^{\lambda-1}, \psi \to \chi \Rightarrow \Delta}$$

Then $d_{\mathcal{H}}$ ends with

$$\frac{\mathcal{H}' \mid \Pi, \psi \Rightarrow \chi}{\mathcal{H}' \mid \Pi \Rightarrow \psi \to \chi}$$

and we can apply (EW) and the induction hypothesis once to obtain

$$\vdash_{\mathrm{GL}^{\circ}} \mathcal{H}' \mid \mathcal{G}'' \mid \Gamma_1, \Pi^{\mu+1} \Rightarrow \chi$$

and again to obtain

$$\vdash_{\mathrm{GL}^{\circ}} \mathcal{H}' \mid \mathcal{G}'' \mid \Gamma_1, \Gamma_2, \Pi^{\lambda} \Rightarrow \Delta. \qquad \square$$

One immediate consequence of this cut elimination result is that cut-free derivations in the considered calculi have the subformula property: any formula occurring in such a derivation must occur as a subformula in the derived hypersequent. The subformula property is crucial not only for developing automated reasoning methods based on Gentzen systems, but also for more theoretical applications. Notice for example that the empty sequent $\Rightarrow$ (which leads to inconsistency in logics with weakening) can, by the subformula property, only be derivable if it is an axiom of the calculus. More generally, a hypersequent $\mathcal{G}$ is derivable iff it is derivable when the logical rules are restricted to those for connectives occurring in $\mathcal{G}$.

Let us make this last claim more precise. A system C_1 for a set of structures Φ_1 is a *conservative extension* of a system C_2 for $\Phi_2 \subseteq \Phi_1$ if for all $a \in \Phi_2$: $\vdash_{C_1} a$ iff $\vdash_{C_2} a$. In this case, C_2 is often called the Φ_2*-fragment* of C_1. By cut elimination, any extension GL of GUL with substitutive single-conclusion rules or GIUL with substitutive rules is a conservative extension of the calculus GL with a restricted set of logical rules. However, this is cheating slightly. Although rules for $\vee$ may not be available in the restricted calculus, we still have plenty of rules for the external disjunction "|": namely, (EW), (EC), and (COM). It is therefore more interesting to ask if we can obtain conservative extension results for Hilbert systems, where only formulas are involved in derivations. Or, put another way, can we find axiomatizations for fragments of Hilbert systems for fuzzy logics? Let us consider a pertinent example: the implicational fragment of monoidal t-norm logic MTL.

The Hilbert system BCK consists of the modus ponens rule (MP) and axiom schema:

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(transitivity)
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(permutation)
(K)	$\varphi \to (\psi \to \varphi)$	(weakening).

Let $\mathrm{HMTL}^{\to}$ be BCK extended with the axiom schema

$$\text{(IPRL)} \quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$

and let $\mathrm{GMTL}^{\to}$ be GMTL with the logical rules restricted to $(\to\Rightarrow)$ and $(\Rightarrow\to)$.

THEOREM 4.1.2. HMTL *is a conservative extension of* $\mathrm{HMTL}^{\to}$.

Proof. Suppose that $\vdash_{\mathrm{HMTL}} \varphi$ for some implicational formula φ. Then making use of Theorems 3.3.2 and 4.1.1, $\vdash_{\mathrm{GMTL}^{\to}} \Rightarrow \varphi$. So it remains to show that $\vdash_{\mathrm{GMTL}^{\to}} \Rightarrow \varphi$ implies $\vdash_{\mathrm{HMTL}^{\to}} \varphi$. We define an interpretation of single-conclusion hypersequents, parameterized by the variable q, that avoids any mention of connectives other than $\to$:

$$\begin{aligned} \mathrm{I_q}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi) &=_{\text{def}} \varphi_1 \to \ldots \to \varphi_n \to \psi \\ \mathrm{I_q}(\varphi_1, \ldots, \varphi_n \Rightarrow) &=_{\text{def}} \varphi_1 \to \ldots \to \varphi_n \to q \\ \mathrm{I_q}(S_1 \mid \ldots \mid S_n) &=_{\text{def}} (\mathrm{I_q}(S_1) \to q) \to \ldots \to (\mathrm{I_q}(S_n) \to q) \to q \end{aligned}$$

where $\varphi_1 \to \ldots \to \varphi_n \to \psi$ is short for $\varphi_1 \to (\varphi_2 \to (\ldots \to (\varphi_n \to \psi)\ldots)$. Observe now that it suffices to prove the following:

If $\vdash_{\mathrm{GMTL}^{\to}} \mathcal{G}$, then $\vdash_{\mathrm{HMTL}^{\to}} \mathrm{I_q}(\mathcal{G})$ for any variable q not occurring in $\mathcal{G}$.

If $\vdash_{\mathrm{GMTL}^{\to}} \Rightarrow \varphi$, then, by the above claim, $\vdash_{\mathrm{HMTL}^{\to}} (\varphi \to q) \to q$ for some q not occurring in φ. But then substituting φ for q in the $\mathrm{HMTL}^{\to}$-derivation, we obtain $\vdash_{\mathrm{HMTL}^{\to}} (\varphi \to \varphi) \to \varphi$ and hence also $\vdash_{\mathrm{HMTL}^{\to}} \varphi$.

We prove the claim by induction on the height of a $\mathrm{GMTL}^{\to\circ}$-derivation of $\mathcal{G}$. For the base case, $\mathcal{G}$ is of the form $\mathcal{G}' \mid \chi \Rightarrow \chi$ and we simply note that $\vdash_{\mathrm{HMTL}^{\to}} \psi \to (((\chi \to \chi) \to q) \to q)$ for any formula ψ. The inductive step involves a number of tedious Hilbert derivations. For rules of the form $(\mathcal{G} \mid S_1) \ldots (\mathcal{G} \mid S_n) \;/\; (\mathcal{G} \mid S)$, it is sufficient to show that:

$$\vdash_{\mathrm{HMTL}^{\to}} \mathrm{I_q}(S_1) \to \ldots \to \mathrm{I_q}(S_n) \to \mathrm{I_q}(S).$$

For example, for $(\to\Rightarrow)$, letting $\Gamma_1 = [\chi_1, \ldots, \chi_k]$ and $\psi = \mathrm{I_q}(\Gamma_2 \Rightarrow \Delta)$, we have a $\mathrm{HMTL}^{\to}$-derivation of

$$(\chi_1 \to \ldots \to \chi_k \to \varphi_1) \to (\varphi_2 \to \psi) \to (\chi_1 \to \ldots \to \chi_k \to (\varphi_1 \to \varphi_2) \to \psi).$$

For cases not of this form, we proceed a little differently. E.g., for (EC), suppose that $\vdash_{\mathrm{GMTL}^{\to}} \mathcal{G} \mid S \mid S$. Then by the induction hypothesis, for some suitable $\chi_1, \ldots, \chi_n$:

$$\vdash_{\mathrm{HMTL}^{\to}} (\chi_1 \to q) \to \ldots (\chi_n \to q) \to (\mathrm{I_q}(S) \to q) \to (\mathrm{I_q}(S) \to q) \to q.$$

Substituting $(\chi_1 \to q) \to \ldots (\chi_n \to q) \to (\mathrm{I_q}(S) \to q) \to q$ for q in the above derivation and simplifying, we obtain as required

$$\vdash_{\mathrm{HMTL}^{\to}} (\chi_1 \to q) \to \ldots (\chi_n \to q) \to (\mathrm{I_q}(S) \to q) \to q. \qquad \square$$

Similar results can be obtained for Hilbert systems for other logics with weakening such as Gödel logic and involutive monoidal t-norm logic. However, in the case of weakening-free logics, we can no longer simulate the connective $\vee$ using $\to$ or the other multiplicative connectives. A particularly interesting open question is to provide an axiomatization (if one exists) of the implicational fragment of HUL. (Could this perhaps be just the implicational fragment of HMAILL?)

For sequent calculi, cut elimination is often a key tool for establishing the decidability of derivability of sequents in the calculus and hence also of derivability of formulas in the corresponding Hilbert system (or indeed, of the equational theory of the corresponding class of algebras). As an easy example, notice that proof search in GMAILL°, proceeding by applying rules backwards, must terminate since the sum of the complexities of formulas in sequents decreases from conclusion to premises in any instance of a rule of this calculus. Hence GMAILL-derivability (also HMAILL-derivability of formulas and the equational theory of FL_e-algebras) is decidable. The same argument works for GMAILL with weakening and related calculi, but fails in the presence of contraction rules where premises can have greater complexity than the conclusion. Sometimes, e.g., for fragments of the relevance logic R, this can be dealt with by using restricted rules and some kind of loop-checking mechanism. For hypersequent calculi, however, the issue is further complicated by the presence of the external contraction rule (EC) which duplicates whole sequents. Nevertheless, for calculi with both internal and external contraction rules, such as GG, GIUML, and GUML, we can again obtain terminating proof search fairly easily.

Let us define a *3-sequent* to be any sequent $\Gamma \Rightarrow \Delta$ such that each formula occurs no more than three times in Γ and no more than three times in Δ, and a *3-hypersequent* to be a hypersequent consisting only of 3-sequents and containing no more than three copies of the same sequent. Now consider the set of formulas F occurring in some 3-hypersequent. There is only a finite number of 3-sequents containing subformulas of the formulas in F, and hence also only a finite number of 3-hypersequents built from such 3-sequents. It is straightforward, using cut elimination, to prove that if GL is GUML, GIUML, or GG, then a 3-hypersequent is GL-derivable iff it has a GL-derivation in which only 3-hypersequents occur. Decidability of derivability of hypersequents in these calculi then follows by restricting to derivations in which only 3-hypersequents occur and using loop-checking (i.e., checking that the premises of a rule instance have not already occurred lower in the derivation).

THEOREM 4.1.3. *Derivability in* GUML*,* GIUML*, and* GG *is decidable.*

Note that the logics MTL, IMTL, and SMTL (and hence also the derivability of hypersequents in GMTL, GIMTL, and GSMTL) have been proved decidable using algebraic methods, while the decidability of the logics UL and IUL (and their corresponding calculi) is still open.

4.2 Density elimination

We turn our attention now to a interesting relative of the cut rule, the hypersequent version of a "density rule" introduced by Takeuti and Titani to axiomatize first-order

Gödel logic:

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow p, \Delta_1 \mid \Gamma_2, p \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \text{ (DENSITY)}$$

where p does not occur in $\mathcal{G}$, Γ_1, Γ_2, Δ_1, or Δ_2.

Note that (DENSITY) does not have the subformula property and cannot be presented schematically without the additional condition on the variable p. Roughly speaking, the rule expresses the "density" of the set of truth values for the logic. Consider an instance $(\varphi \Rightarrow p \mid p \Rightarrow \psi) \mathbin{/} \varphi \Rightarrow \psi$. Understanding $\mid$ as a classical disjunction and $\Rightarrow$ as $\leq$, we can read such an instance contrapositively as "if $\varphi > \psi$, then $\varphi > p > \psi$ for some p".

For a hypersequent calculus GL, let GL^{D} be GL extended with (DENSITY). Our goal in this section will be to show that for certain (families of) hypersequent calculi GL, the extended calculus GL^{D} admits *density elimination*; in other words, we can add (DENSITY), but we don't really need it.

EXAMPLE 4.2.1. *Adding* (DENSITY) *can have a dramatic effect. If* GL *is a calculus with* (ID), (COM), *and* (C) *(without single-conclusion restrictions), then the empty sequent is derivable in* GL^{D} *as follows:*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{p \Rightarrow p}\text{ (ID)} \quad \cfrac{}{p \Rightarrow p}\text{ (ID)}}{\Rightarrow p, p \mid p, p \Rightarrow}\text{ (COM)}}{\Rightarrow p, p \mid p \Rightarrow}\text{ (C)}}{\Rightarrow p \mid p \Rightarrow}\text{ (C)}}{\Rightarrow}\text{ (DENSITY)}$$

In particular, adding (DENSITY) *to a hypersequent calculus for classical logic gives inconsistency. Moreover,* (DENSITY) *cannot be eliminated from the calculus* GRM^{D} *since the empty sequent is derivable in* GRM^{D} *but not in* GRM.

Our density elimination method will proceed – like cut elimination – by removing applications of the rule which are uppermost in a derivation. Suppose that we have a derivation d ending with

$$\cfrac{\cfrac{\vdots}{\Gamma \Rightarrow p \mid \Pi, p \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Sigma}\text{ (DENSITY)}$$

The idea of our proof is to replace occurrences of p in d in an "asymmetric" way: with Γ if p occurs on the left, and with Π on the left and Σ on the right, if p occurs on the right. What we get is not quite a derivation, but still a finite tree labelled with hypersequents, now ending with

$$\cfrac{\cfrac{\vdots}{\Gamma, \Pi \Rightarrow \Sigma \mid \Pi, \Gamma \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Sigma}$$

The last step in this not-quite-a-derivation is an application of (EC). The applications of logical rules and most structural rules appearing in the original derivation are preserved by the replacement due to substitutivity or almost-substitutivity. Where the derivation potentially breaks down is in rules like (COM) where ps can occur in premises on both the left and the right. For example, suppose that d ends with

$$\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\ (\mathrm{ID}) \quad \dfrac{\vdots}{\Gamma', \Pi \Rightarrow \Sigma}}{\Gamma' \Rightarrow p \mid \Pi, p \Rightarrow \Sigma}\ (\mathrm{COM})}{\vdots}$$

$$\dfrac{\Gamma \Rightarrow p \mid \Pi, p \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma}\ (\mathrm{DENSITY})$$

If we replace ps as suggested, we get

$$\dfrac{\dfrac{\Gamma, \Pi \Rightarrow \Sigma \quad \dfrac{\vdots}{\Gamma', \Pi \Rightarrow \Sigma}}{\Gamma', \Pi \Rightarrow \Sigma \mid \Gamma, \Pi \Rightarrow \Sigma}\ (\mathrm{COM})}{\vdots}$$

$$\dfrac{\Gamma, \Pi \Rightarrow \Sigma \mid \Gamma, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma}\ (\mathrm{EC})$$

But now we have a missing part of the derivation: the sub-derivation of $(\Gamma, \Pi \Rightarrow \Sigma)$, which was what we wanted to prove in the first place. However, notice that in this case, we can simply replace the application of (COM) with an application of (EW) and remove the occurrence of $(\Gamma, \Pi \Rightarrow \Sigma)$ as a premise. More generally, we are able to use (CUT) and cut elimination to repair such derivations.

Let us assume for now that we deal only with single-conclusion hypersequents. Moreover, we consider hypersequents where the variable p occurs only in a limited fashion: only as an atom and not on both the left and right in the same sequent. More precisely, a hypersequent $\mathcal{G}$ is called *p-regular* if it is of the form

$$[\Gamma_i \Rightarrow p]_{i=1}^{n} \mid [\Pi_j, [p]^{\lambda_j} \Rightarrow \Sigma_j]_{j=1}^{m}$$

where p does not occur in $\Gamma_1, \ldots, \Gamma_n, \Pi_1, \ldots, \Pi_m, \Sigma_1, \ldots, \Sigma_m$.

We also need a way of distinguishing the occurrences of the variable p introduced by the density rule. A *double-p-marked hypersequent* has just two occurrences of p, both marked: one on the left in a sequent and one on the right in another sequent, written

$$\mathcal{H} \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma.$$

The idea is to combine a p-regular hypersequent $\mathcal{G}$ with a double-p-marked hypersequent $(\mathcal{H} \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma)$, essentially by applying (CUT) exhaustively to $\mathcal{G}$ with $(\mathcal{H} \mid \Gamma \Rightarrow \underline{p})$ and $(\mathcal{H} \mid \Pi, \underline{p} \Rightarrow \Sigma)$ with cut formula p. Suppose that

1. $\mathcal{G} = [\Gamma_i \Rightarrow p]_{i=1}^{n} \mid [\Pi_j, [p]^{\lambda_j} \Rightarrow \Sigma_j]_{j=1}^{m}$ is a p-regular hypersequent
2. $\mathcal{H}_p = (\mathcal{H} \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma)$ is a double-p-marked hypersequent.

Then $\mathrm{DEN}(\mathcal{G}, \mathcal{H}_p) = (\mathcal{H} \mid [\Gamma_i, \Pi \Rightarrow \Sigma]_{i=1}^{n} \mid [\Pi_j, \Gamma^{\lambda_j} \Rightarrow \Sigma_j]_{j=1}^{m})$.

EXAMPLE 4.2.2. *Consider the p-regular and double-p-marked hypersequents*

$$\mathcal{G} = (q \to r \Rightarrow p \mid q \Rightarrow p \mid r \to q, p, p \Rightarrow r) \quad \textit{and} \quad \mathcal{H}_p = (s \Rightarrow \underline{p} \mid s \to q, \underline{p} \Rightarrow r).$$

Then $\mathrm{DEN}(\mathcal{G}, \mathcal{H}_p) = (q \to r, s \to q \Rightarrow r \mid q, s \to q \Rightarrow r \mid r \to q, s, s \Rightarrow r)$.

Note that a double-p-marked hypersequent $\mathcal{H}_p = (\mathcal{H} \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma)$ is always p-regular and

$$\mathrm{DEN}(\mathcal{H}_p, \mathcal{H}_p) = (\mathcal{H} \mid \mathcal{H} \mid \Gamma, \Pi \Rightarrow \Sigma \mid \Gamma, \Pi \Rightarrow \Sigma).$$

Now let us apply these ideas to a concrete family of calculi. We will call a hypersequent rule *local* if it is the hypersequent version of a sequent rule, substitutive, and for each of its instances, any variable occurring on the left (right) in a premise must occur on the left (right) in the conclusion. In other words, local rules do not shift variables from one side of a sequent to the other. The structural rules (W), (C), (MIX), (EMP), and ($\mathrm{C_n}$) ($n \geq 2$) are all local, but not, for example

$$\frac{\mathcal{G} \mid \Delta \Rightarrow \Gamma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}$$

THEOREM 4.2.3. *Let* GL *be any extension of* GMTL *with local single-conclusion rules. Then* $\mathrm{GL^D}$ *admits density elimination.*

Proof. As for cut elimination, it is enough to consider the uppermost applications of (DENSITY). More generally, we prove that for any p-regular hypersequent $\mathcal{G}$ and double-p-marked hypersequent $\mathcal{H}_p = (\mathcal{H} \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma)$:

If $d \vdash_{\mathrm{GL}^\circ} \mathcal{G}$ and $d' \vdash_{\mathrm{GL}^\circ} \mathcal{H}_p$, then $\vdash_{\mathrm{GL}} \mathrm{DEN}(\mathcal{G}, \mathcal{H}_p)$.

To see that this suffices, consider an uppermost application of (DENSITY) with premise $\mathcal{G} = (\mathcal{G}' \mid \Gamma \Rightarrow p \mid \Pi, p \Rightarrow \Sigma)$. Suppose that $\vdash_{\mathrm{GL}} \mathcal{G}$. Then by cut elimination, $\vdash_{\mathrm{GL}^\circ} \mathcal{G}$ and it follows from the claim with $\mathcal{H}_p = (\mathcal{G}' \mid \Gamma \Rightarrow \underline{p} \mid \Pi, \underline{p} \Rightarrow \Sigma)$ that $\vdash_{\mathrm{GL}} \mathcal{G}' \mid \mathcal{G}' \mid \Gamma, \Pi \Rightarrow \Sigma \mid \Gamma, \Pi \Rightarrow \Sigma$. So by (EC), $\vdash_{\mathrm{GL}} \mathcal{G}' \mid \Gamma, \Pi \Rightarrow \Sigma$ as required.

We prove the claim by induction on $\mathrm{h}(d)$. Consider the last rule (r) applied in d. If (r) is (ID) and $\mathcal{G} = (\mathcal{G}' \mid \varphi \Rightarrow \varphi)$ (where φ cannot be p since $\mathcal{G}$ is p-regular), then $\mathrm{DEN}(\mathcal{G}, \mathcal{H}_p) = (\mathcal{H}' \mid \varphi \Rightarrow \varphi)$ for some $\mathcal{H}'$ and is derivable by (ID). If (r) is (EC) or (EW), then the claim follows by applying the induction hypothesis and (r). Otherwise:

- Suppose that (r) is a rule other than (ID), (EC), (EW), or (COM), and d ends with

  $$\frac{\mathcal{G}' \mid S_1 \ \ldots \ \mathcal{G}' \mid S_n}{\mathcal{G}' \mid S} \ (r)$$

  Note that $(\mathcal{G}' \mid S)$ is p-regular by assumption. Also (r) is local so occurrences of p cannot "switch sides" in a sequent from the premises to the conclusion. Hence $(\mathcal{G}' \mid S_1), \ldots, (\mathcal{G}' \mid S_n)$ are all p-regular, and by the induction hypothesis:

  $$\vdash_{\mathrm{GL}} \mathrm{DEN}((\mathcal{G}' \mid S_i), \mathcal{H}_p) \quad \text{for } i = 1 \ldots n.$$

But $\text{DEN}((\mathcal{G}' \mid S), \mathcal{H}_p)$ is the result of multiple applications of (CUT) between $(\mathcal{H} \mid \mathcal{G}' \mid S)$ and $(\Gamma \Rightarrow \underline{p})$ and $(\Pi, \underline{p} \Rightarrow \Sigma)$. Since (r) is substitutive or almost-substitutive and p cannot occur in the premises of an instance of (r) with no p in the conclusion, we obtain an instance of (r)

$$\frac{\text{DEN}((\mathcal{G}' \mid S_1), \mathcal{H}_p) \quad \ldots \quad \text{DEN}((\mathcal{G}' \mid S_n), \mathcal{H}_p)}{\text{DEN}((\mathcal{G}' \mid S), \mathcal{H}_p)}$$

So $\vdash_{\text{GL}} \text{DEN}((\mathcal{G}' \mid S), \mathcal{H}_p)$ as required.

- Suppose now that (r) is (COM). If both premises are p-regular, then the claim follows by applying the induction hypothesis to the premises and using (COM). For example, suppose that d ends with:

$$\frac{\mathcal{G}' \mid \Gamma_1, \Pi_1 \Rightarrow p \quad \mathcal{G}' \mid \Gamma_2, \Pi_2, [p]^k \Rightarrow \Delta}{\mathcal{G}' \mid \Gamma_1, \Gamma_2, [p]^k \Rightarrow \Delta \mid \Pi_1, \Pi_2 \Rightarrow p} \ (\text{COM})$$

By the induction hypothesis twice:

$\vdash_{\text{GL}} \text{DEN}(\mathcal{G}', \mathcal{H}_p) \mid \Gamma_1, \Pi_1, \Pi \Rightarrow \Sigma$ and $\vdash_{\text{GL}} \text{DEN}(\mathcal{G}', \mathcal{H}_p) \mid \Gamma_2, \Pi_2, \Gamma^k \Rightarrow \Delta$.

Hence by (COM), as required:

$$\vdash_{\text{GL}} \text{DEN}(\mathcal{G}', \mathcal{H}_p) \mid \Gamma_1, \Gamma_2, \Gamma^k \Rightarrow \Delta \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma.$$

Suppose then that one of the premises is not p-regular and d ends with:

$$\frac{\mathcal{G}' \mid \Gamma_1, \Pi_1, [p]^{m+1} \Rightarrow p \quad \mathcal{G}' \mid \Gamma_2, [p]^k, \Pi_2 \Rightarrow \Delta}{\mathcal{G}' \mid \Gamma_1, \Gamma_2, [p]^{k+m+1} \Rightarrow \Delta \mid \Pi_1, \Pi_2 \Rightarrow p} \ (\text{COM})$$

Let $\mathcal{G}_1 = \text{DEN}(\mathcal{G}', \mathcal{H}_p)$. Then by the induction hypothesis:

$$d_1 \vdash_{\text{GL}} \mathcal{G}_1 \mid \Gamma_2, \Gamma^k, \Pi_2 \Rightarrow \Delta.$$

Our aim is to find a derivation for

$$\vdash_{\text{GL}} \mathcal{G}_1 \mid \Gamma_1, \Gamma_2, \Gamma^{k+m+1} \Rightarrow \Delta \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma.$$

Consider the GL°-derivation d' of $(\mathcal{H} \mid \Gamma \Rightarrow p \mid \Pi, p \Rightarrow \Sigma)$. We can substitute $\&\Pi_2$ (recalling that $\&[] = \overline{1}$) for p in this derivation to get

$$d_2 \vdash_{\text{GL}^\circ} \mathcal{H} \mid \Gamma \Rightarrow \&\Pi_2 \mid \Pi, \&\Pi_2 \Rightarrow \Sigma.$$

Let d_3 be the (easy) derivation of $(\mathcal{H} \mid \Pi_2 \Rightarrow \&\Pi_2)$ using $(\Rightarrow\&)$, $(\Rightarrow \overline{1})$, and (ID), and let d_2' be the derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots\, d_2}{\mathcal{H} \mid \Gamma \Rightarrow \&\Pi_2 \mid \Pi, \&\Pi_2 \Rightarrow \Sigma}}{\mathcal{H} \mid \Gamma^{m+1} \Rightarrow \&\Pi_2 \mid \Pi, \&\Pi_2 \Rightarrow \Sigma}\ (\text{W})}{\mathcal{H} \mid \Gamma^{m+1} \Rightarrow \&\Pi_2 \mid \Pi_1, \&\Pi_2, \Pi \Rightarrow \Sigma}\ (\text{W}) \qquad \cfrac{\vdots\, d_3}{\mathcal{H} \mid \Pi_2 \Rightarrow \&\Pi_2}}{\cfrac{\mathcal{H} \mid \Gamma^{m+1} \Rightarrow \&\Pi_2 \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma}{\mathcal{G}_1 \mid \Gamma^{m+1} \Rightarrow \&\Pi_2 \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma}\ (\text{EW})}\ (\text{CUT})$$

Also, let d_1' be the derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots\, d_1}{\mathcal{G}_1 \mid \Gamma_2, \Gamma^k, \Pi_2 \Rightarrow \Delta}}{\vdots}\ {\scriptstyle (\&\Rightarrow)\text{ or }(\bar{1}\Rightarrow)}}{\mathcal{G}_1 \mid \Gamma_2, \Gamma^k, \&\Pi_2 \Rightarrow \Delta}\ {\scriptstyle (\&\Rightarrow)\text{ or }(\bar{1}\Rightarrow)}}{\mathcal{G}_1 \mid \Gamma_1, \Gamma_2, \Gamma^k, \&\Pi_2 \Rightarrow \Delta}\ {\scriptstyle (\mathrm{W})}$$

Finally, putting these pieces together, we obtain the required derivation:

$$\cfrac{\cfrac{\vdots\, d_1'}{\mathcal{G}_1 \mid \Gamma_1, \Gamma_2, \Gamma^k, \&\Pi_2 \Rightarrow \Delta} \quad \cfrac{\vdots\, d_2'}{\mathcal{G}_1 \mid \Gamma^{m+1} \Rightarrow \&\Pi_2 \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma}}{\mathcal{G}_1 \mid \Gamma_1, \Gamma_2, \Gamma^{k+m+1} \Rightarrow \Delta \mid \Pi_1, \Pi_2, \Pi \Rightarrow \Sigma}\ {\scriptstyle (\mathrm{CUT})^*}$$

□

COROLLARY 4.2.4. $\mathrm{GMTL^D}$, $\mathrm{GSMTL^D}$, $\mathrm{GG^D}$, *and* $\mathrm{GMTL_n^D}$ *(*$n \geq 3$*) admit density elimination.*

While for single-conclusion calculi with weakening, density elimination goes hand-in-hand with cut elimination, calculi that lack weakening and/or are multiple-conclusion, are more difficult to deal with. Density elimination has been established for several such calculi, in particular, GIMTL for involutive monoidal t-norm logic and GUL for uninorm logic (or semilinear bounded FL_e-algebras), making use of tailored induction hypotheses. However, no method is yet known for GIUL.

THEOREM 4.2.5. $\mathrm{GIMTL^D}$, $\mathrm{GNM^D}$, $\mathrm{GUL^D}$, $\mathrm{GUML^D}$, *and* $\mathrm{GIUML^D}$ *admit density elimination.*

The key application of density elimination is to establish standard completeness results for fuzzy logics, or, equivalently, to show that certain varieties of semilinear bounded FL_e-algebras are generated (as quasivarieties) by their members with lattice reduct $[0, 1]$. More precisely, density elimination can be used to show that a fuzzy logic is complete with respect to a class of linearly and densely ordered bounded FL_e-algebras (so-called "rational completeness"); completeness with respect to a class of bounded FL_e-algebras with lattice reduct $[0, 1]$ is then established algebraically using a Dedekind-MacNeille completion procedure.

Recall (e.g., from previous chapters of this handbook) that with any axiomatic extension HL of HUL, we can associate a variety $\mathbb{L}$ of bounded FL_e-algebras, having linearly ordered members (chains) $\mathbb{LIN}(\mathbb{L})$, in such a way that for all $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_p}$:

$$T \vdash_{\mathrm{HL}} \varphi \quad \text{iff} \quad \{\bar{1} \leq \psi \mid \psi \in T\} \models_{\mathbb{L}} \bar{1} \leq \varphi \quad \text{iff} \quad \{\bar{1} \leq \psi \mid \psi \in T\} \models_{\mathbb{LIN}(\mathbb{L})} \bar{1} \leq \varphi.$$

Let us now define $\mathbb{DEN}(\mathbb{L})$ to be the class of all dense linearly ordered members of $\mathbb{L}$, and say that HL has the *density property* if for any $T \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}_p}$ and variable p not occurring in $T \cup \{\varphi, \psi, \chi\}$:

$$T \vdash_{\mathrm{HL}} (\varphi \to p) \vee (p \to \psi) \vee \chi \quad \Rightarrow \quad T \vdash_{\mathrm{HL}} (\varphi \to \psi) \vee \chi.$$

A proof of the following lemma (in a more general setting) may be found in Chapter II of this handbook:

LEMMA 4.2.6. *For any axiomatic extension* HL *of* HUL *with the density property:*

$$T \vdash_{\mathrm{HL}} \varphi \quad \text{iff} \quad \{\overline{1} \leq \psi \mid \psi \in T\} \models_{\mathbb{DEN}(\mathbb{L})} \overline{1} \leq \varphi.$$

Suppose now that there is a hypersequent calculus GL that is sound and complete with respect to HL such that GL^{D} admits density elimination. By the local deduction theorem for axiomatic extensions of HUL (also established in a more general setting in Chapter II of this handbook), if $T \vdash_{\mathrm{HL}} (\varphi \to p) \vee (p \to \psi) \vee \chi$, then for some $\varphi^* = (\varphi_1 \wedge \overline{1})$ & $\ldots$ & $(\varphi_m \wedge \overline{1})$ with $\{\varphi_1, \ldots, \varphi_m\} \subseteq T$

$$\vdash_{\mathrm{HL}} \varphi^* \to ((\varphi \to p) \vee (p \to \psi) \vee \chi)$$

and, using some derivabilities in HUL

$$\vdash_{\mathrm{HL}} ((\varphi^* \mathbin{\&} \varphi) \to p) \vee (p \to (\varphi^* \to \psi)) \vee (\varphi^* \to \chi).$$

But then by the completeness of GL

$$\vdash_{\mathrm{GL}} \varphi^*, \varphi \Rightarrow p \mid p, \varphi^* \Rightarrow \psi \mid \varphi^* \Rightarrow \chi$$

and using first the density rule and then density elimination for GL^{D}

$$\vdash_{\mathrm{GL}} \varphi^*, \varphi, \varphi^* \Rightarrow \psi \mid \varphi^* \Rightarrow \chi.$$

Finally, using the soundness of GL and some derivabilities in HUL

$$\vdash_{\mathrm{HL}} (\varphi^* \mathbin{\&} \varphi^*) \to ((\varphi \to \psi) \vee \chi)$$

and so by the other direction of the local deduction theorem

$$T \vdash_{\mathrm{HL}} (\varphi \to \psi) \vee \chi.$$

Hence HL has the density property, and we obtain, using Lemma 4.2.6, Corollary 4.2.4, and Theorem 4.2.5:

THEOREM 4.2.7. *If* L *is* MTL, SMTL, $\mathrm{MTL_n}$ *(*$n \geq 3$*)*, GG, IMTL, NM, UL, UML, *or* IUML, *then*

$$T \vdash_{\mathrm{HL}} \varphi \quad \text{iff} \quad \{\overline{1} \leq \psi \mid \psi \in T\} \models_{\mathbb{DEN}(\mathbb{L})} \overline{1} \leq \varphi.$$

5 The fundamental fuzzy logics

We turn our attention in this section to developing proof theory for some of the most renowned – or in Hájek's parlance, fundamental – fuzzy logics: Gödel logic G, Łukasiewicz logic Ł, and product logic P. For Gödel logic, we introduce and exploit some interesting alternatives to the hypersequent calculus GG. For Łukasiewicz logic and product logic, where the standard approach described in the last two sections fails, we obtain calculi by changing the interpretation of hypersequents and introducing new logical rules. Finally, we consider a generalization of hypersequents for which these three logics have a uniform proof-theoretic presentation.

5.1 Gödel logic

Let us first refresh our memory of Gödel logic G in the language $\mathcal{L}_c$ with binary connectives $\wedge, \vee, \to$ and constants $\bot, \top$. The standard semantics of G is characterized by the Gödel t-norm $\min$ and its residuum $\to_{\mathrm{G}}$, defined on the real unit interval $[0, 1]$ by

$$x \to_{\mathrm{G}} y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$$

More precisely, a G-*valuation* is a function $v\colon Fm_{\mathcal{L}_c} \to [0, 1]$ satisfying

$$\begin{array}{rcl} v(\bot) & = & 0 \\ v(\top) & = & 1 \\ v(\varphi \wedge \psi) & = & \min(v(\varphi), v(\psi)) \\ v(\varphi \vee \psi) & = & \max(v(\varphi), v(\psi)) \\ v(\varphi \to \psi) & = & v(\varphi) \to_{\mathrm{G}} v(\psi). \end{array}$$

A formula φ is G-*valid*, written $\models_{\mathrm{G}} \varphi$, iff $v(\varphi) = 1$ for all G-valuations v.

The hypersequent calculus GG is an elegant and informative presentation of Gödel logic, an extension both of GMTL (for monoidal t-norm logic) with contraction, and of a hypersequent version of Gentzen's LJ (for intuitionistic logic) with communication. Cut elimination for GG provides an easy proof of decidability, and, via density elimination, a more complicated proof of standard completeness. However, even with loop-checking, which gives termination of the rules (read upwards), GG is not particularly efficient for theorem proving. External contraction can double the size of hypersequents, and since not all the logical rules of the calculus are invertible, backtracking is required for proof search. Below we consider two calculi that address these issues, starting with a system in Gentzen's original sequent framework.

Since sequents are not as flexible as hypersequents, we remove the restriction to single-conclusion sequents and define (recalling that $\bigwedge[] =_{\text{def}} \top$ and $\bigvee[] =_{\text{def}} \bot$):

$$\mathrm{I}_{\mathrm{G}}(\Gamma \Rightarrow \Delta) =_{\text{def}} \bigwedge \Gamma \to \bigvee \Delta.$$

We also make use of more complicated rules for connectives that "decompose" formulas into formulas with a smaller complexity, assuming for convenience that $\varphi \vee \psi =_{\text{def}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ since the number of rules required increases exponentially with the number of connectives.

Let us call a sequent *atomic implicational* if it contains only atoms and implications of the form $a \to b$ where a and b are atoms. Then it is easily seen that (read upwards) the rules defined in Figure 5 reduce any sequent to a set of atomic implicational sequents. Moreover, by simple arithmetic, these rules are sound and invertible with respect to G-validity (i.e., the conclusion of each rule instance is G-valid iff all the premises are G-valid). So we can check if a sequent is G-valid by applying the decomposition rules (upwards) exhaustively, then checking the G-validity of the resulting atomic implicational sequents. For this last step, we may also give a more immediately meaningful presentation of G-validity.

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\Rightarrow)_{\mathrm{G}} \qquad \frac{\Gamma, \varphi \to \chi \Rightarrow \Delta \quad \Gamma, \psi \to \chi \Rightarrow \Delta}{\Gamma, \varphi \to (\psi \to \chi) \Rightarrow \Delta}\ (\to(\to)\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ (\Rightarrow\wedge) \qquad \frac{\Gamma, \psi \to \chi \Rightarrow \varphi \to \psi, \Delta \quad \Gamma, \chi \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \chi \Rightarrow \Delta}\ ((\to)\to\Rightarrow)$$

$$\frac{\Gamma, \varphi \to \psi, \varphi \to \chi \Rightarrow \Delta}{\Gamma, \varphi \to (\psi \wedge \chi) \Rightarrow \Delta}\ (\to\wedge\Rightarrow) \qquad \frac{\Gamma, \varphi \to \chi \Rightarrow \Delta \quad \Gamma, \psi \to \chi \Rightarrow \Delta}{\Gamma, (\varphi \wedge \psi) \to \chi \Rightarrow \Delta}\ (\wedge\to\Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi \to \chi, \psi \to \chi, \Delta}{\Gamma \Rightarrow (\varphi \wedge \psi) \to \chi, \Delta}\ (\Rightarrow\wedge\to) \qquad \frac{\Gamma \Rightarrow \varphi \to \psi, \Delta \quad \Gamma \Rightarrow \varphi \to \chi, \Delta}{\Gamma \Rightarrow \varphi \to (\psi \wedge \chi), \Delta}\ (\Rightarrow\to\wedge)$$

$$\frac{\Gamma \Rightarrow \varphi \to \chi, \psi \to \chi, \Delta}{\Gamma \Rightarrow \varphi \to (\psi \to \chi), \Delta}\ (\Rightarrow\to(\to)) \qquad \frac{\Gamma \Rightarrow \psi \to \chi, \Delta \quad \Gamma, \varphi \to \psi \Rightarrow \chi, \Delta}{\Gamma \Rightarrow (\varphi \to \psi) \to \chi, \Delta}\ (\Rightarrow(\to)\to)$$

Figure 5. Sequent decomposition rules for Gödel logic

An *inequality* is an ordered triple $a \lhd b$ where a and b are atoms and $\lhd \in \{<, \leq\}$. A set of inequalities $\mathcal{S}$ is G-*valid*, written $\models_{\mathrm{G}} \mathcal{S}$, iff for all G-valuations v, $v(a) \lhd v(b)$ for some $a \lhd b \in \mathcal{S}$. Moreover, we define a set of inequalities $\mathrm{Iqs}(\Gamma \Rightarrow \Delta)$ for each atomic implicational sequent $\Gamma \Rightarrow \Delta$ by

$$\begin{array}{llllll} (a \leq b) \in \mathrm{Iqs}(\Gamma \Rightarrow \Delta) & \text{if} & (a \to b) \in \Delta & (\top \leq b) \in \mathrm{Iqs}(\Gamma \Rightarrow \Delta) & \text{if} & b \in \Delta \\ (a < b) \in \mathrm{Iqs}(\Gamma \Rightarrow \Delta) & \text{if} & (b \to a) \in \Gamma & (a < \top) \in \mathrm{Iqs}(\Gamma \Rightarrow \Delta) & \text{if} & a \in \Gamma. \end{array}$$

LEMMA 5.1.1. $\models_{\mathrm{G}} \mathrm{I}_{\mathrm{G}}(\Gamma \Rightarrow \Delta)$ *iff* $\models_{\mathrm{G}} \mathrm{Iqs}(\Gamma \Rightarrow \Delta)$.

Proof. Making use of the standard deduction theorem for G, $\models_{\mathrm{G}} \mathrm{I}_{\mathrm{G}}(\Gamma \Rightarrow \Delta)$ iff for every G-valuation v: either $v(\varphi) < 1$ for some $\varphi \in \Gamma$ or $v(\psi) = 1$ for some $\psi \in \Delta$. So $\models_{\mathrm{G}} \mathrm{I}_{\mathrm{G}}(\Gamma \Rightarrow \Delta)$ iff either $v(a \to b) = 1$ for some $(a \to b) \in \Delta$, $v(b) = 1$ for some $b \in \Delta$, $v(b \to a) < 1$ for some $(b \to a) \in \Gamma$, or $v(a) < 1$ for some $a \in \Gamma$. Hence $\models_{\mathrm{G}} \mathrm{I}_{\mathrm{G}}(\Gamma \Rightarrow \Delta)$ iff $v(a) \lhd v(b)$ for some $(a \lhd b) \in \mathrm{Iqs}(\Gamma \Rightarrow \Delta)$ as required. □

Examples of G-valid sets of inequalities include

$$\{(p \leq q), (q < r), (r \leq s), (s < p)\} \qquad \{(\bot < p), (p < q), (q \leq r)\}.$$

In the first case we have a sequence of inequalities beginning and ending with p, and in the second we have a sequence beginning with $\bot$ and involving a non-strict inequality. This "chain-like" form is a common feature of all G-valid sets of inequalities.

LEMMA 5.1.2. *A set of inequalities* $\mathcal{S}$ *is* G-*valid iff there exists* $(a_i \lhd_i a_{i+1}) \in \mathcal{S}$ *for* $i = 1 \ldots n$ *such that either (1)* $a_1 = a_{n+1}$ *or* $a_1 = \bot$ *or* $a_{n+1} = \top$, *where* $\lhd_i$ *is* $\leq$ *for some* $i \in \{1, \ldots, n\}$, *or (2)* $a_1 = \bot$ *and* $a_{n+1} = \top$.

Proof. $\mathcal{S}$ is clearly G-valid if conditions (1) or (2) are met. For the other direction, we proceed by induction on the number of different variables occurring in $\mathcal{S}$. Note first that if one of $a \leq a$, $a \leq \top$, $\bot \leq a$, or $\bot < \top$ occurs in $\mathcal{S}$, then we are done. This takes

Axioms

$$\frac{}{\Gamma,\varphi \Rightarrow \varphi,\Delta}\ (\textsc{idw}) \qquad \frac{}{\Gamma,\bot \Rightarrow \Delta}\ (\bot\Rightarrow) \qquad \frac{}{\Gamma \Rightarrow \top,\Delta}\ (\Rightarrow\top)$$

Decomposition rules: Figure 5.

Atomic rules

$$\frac{\Gamma, b \Rightarrow \Delta \quad \Gamma \Rightarrow a, \Delta}{\Gamma, a \to b \Rightarrow \Delta}\ (\to\Rightarrow)_a \qquad \frac{\Gamma, a \Rightarrow b}{\Gamma \Rightarrow a \to b, \Delta}\ (\Rightarrow\to)_a \qquad \frac{\Gamma, b \to a \Rightarrow a \to b, \Delta}{\Gamma \Rightarrow a \to b, \Delta}\ (\textsc{lin})$$

Figure 6. The sequent calculus $\mathrm{GG_g}$

care of the cases with at most one variable. Otherwise, we fix a variable q occurring in $\mathcal{S}$, and define

$$\begin{aligned}
\mathcal{S}_< &=_{\text{def}} \{a < b \mid \{a < q, q < b\} \subseteq \mathcal{S}, a \neq q, b \neq q\}\\
\mathcal{S}_\leq &=_{\text{def}} \{a \leq b \mid \{a \lhd_1 q, q \lhd_2 b\} \subseteq \mathcal{S}, a \neq q, b \neq q, \leq \in \{\lhd_1, \lhd_2\}\}\\
\mathcal{S}' &=_{\text{def}} \{a \lhd b \in \mathcal{S} \mid a \neq q, b \neq q\} \cup \mathcal{S}_< \cup \mathcal{S}_\leq.
\end{aligned}$$

$\mathcal{S}'$ has fewer different variables than $\mathcal{S}$. So if $\mathcal{S}'$ is G-valid, then applying the induction hypothesis to $\mathcal{S}'$, we have $(a_i \lhd_i a_{i+1}) \in \mathcal{S}'$ for $i = 1 \ldots n$, satisfying either (1) or (2) above. But then easily by replacing the inequalities $a_i \lhd_i a_{i+1}$ that occur in $\mathcal{S}_<$ or $\mathcal{S}_\leq$ appropriately by $a_i \lhd' q$ and $q \lhd'' a_{i+1}$, we get that (1) or (2) holds for $\mathcal{S}$. Hence it is sufficient to show that $\mathcal{S}'$ is G-valid. Suppose otherwise, i.e., that there exists a G-valuation v such that $v(a) \lhd v(b)$ does not hold for any $a \lhd b \in \mathcal{S}'$. We show for a contradiction that $\mathcal{S}$ is not G-valid. Let

$$x = \min\{v(a) \mid a \lhd q \in \mathcal{S}\} \quad \text{and} \quad y = \max\{v(b) \mid q \lhd b \in \mathcal{S}\}.$$

Note first that $x \geq y$. Otherwise it follows that for some a, b, we have $\{a \lhd_1 q, q \lhd_2 b\} \subseteq \mathcal{S}$ and $v(a) < v(b)$. But then $(a \lhd b) \in \mathcal{S}'$ so $v(a) \geq v(b)$, a contradiction. So there are two cases. If $x > y$, then we change v so that $x > v(q) > y$. For any $(a \lhd q)$ or $(q \lhd b)$ in $\mathcal{S}$, we have $v(a) \geq x > v(q) > y \geq v(b)$. Hence $\mathcal{S}$ is not G-valid, a contradiction. Now suppose that $x = y$ and let us change v so that $v(q) = x$. We must have atoms a_0, b_0 such that $(a_0 < q)$ and $(q < b_0)$ are in $\mathcal{S}$ and $v(a_0) = v(b_0) = v(q)$. Consider any $(a \lhd_1 q)$ or $(q \lhd_2 b)$ in $\mathcal{S}$. Since $(a \lhd_1 b_0)$ and $(a_0 \lhd_2 b)$ are in $\mathcal{S}'$, $v(a) \lhd_1 v(q) = v(b_0)$ and $v(a_0) = v(q) \lhd_2 v(b)$ cannot hold. So $\mathcal{S}$ is again not G-valid, a contradiction. □

The sequent calculus $\mathrm{GG_g}$ displayed in Figure 6 extends the decomposition rules for G with rules for dealing with atomic implicational sequents. The rule $(\to\Rightarrow)_a$ is the usual classical implication left rule restricted to atoms, while $(\Rightarrow\to)_a$ combines applications of weakening and implication right rules. The rule (LIN) is what really extends the calculus beyond intuitionistic logic and characterizes the linearity of the truth values.

EXAMPLE 5.1.3. *We illustrate* $\mathrm{GG_g}$ *with the following derivation:*

$$\frac{\dfrac{}{r \Rightarrow r, q \to p}\ (\textsc{idw}) \qquad \dfrac{\dfrac{}{q \to r, p \to q \Rightarrow p \to q, r, q \to p}\ (\textsc{idw})}{q \to r \Rightarrow p \to q, r, q \to p}\ (\textsc{lin})}{(p \to q) \to r \Rightarrow r, q \to p}\ ((\to)\to\Rightarrow)$$

THEOREM 5.1.4. $\vdash_{\mathrm{GG_g}} S$ *iff* $\models_{\mathrm{G}} \mathrm{I_G}(S)$.

Proof. The left to right direction proceeds as usual by induction on the height of a proof of S in $\mathrm{GG_g}$. For the right to left direction, suppose that $\models_{\mathrm{G}} \mathrm{I_G}(\Gamma \Rightarrow \Delta)$. We can assume that $\Gamma \Rightarrow \Delta$ is atomic implicational. Let $\Gamma' = \Gamma \uplus [a \to b \mid b \to a \in \Delta]$. Observe that $\models_{\mathrm{G}} \mathrm{I_G}(\Gamma' \Rightarrow \Delta)$ and $\Gamma \Rightarrow \Delta$ is derivable from $\Gamma' \Rightarrow \Delta$ using repeated applications of (LIN). Hence there exists a sequence of inequalities $(a_i \lhd_i a_{i+1}) \in \mathrm{Iqs}(\Gamma' \Rightarrow \Delta)$ for $i = 1 \ldots n$ satisfying either (1) or (2) from Lemma 5.1.2. Moreover, we can assume that $\lhd_i$ is $\leq$ for at most one i. Otherwise, replacing any one of the two or more occurrences of $\leq$ with $<$ gives a sequence of inequalities that still satisfies either (1) or (2). If $a \leq b$ is replaced with $a < b$ where $(a \to b) \in \Delta$, then $(b \to a) \in \Gamma$. Also if $b \in \Delta$, then removing $\top \leq b$ still gives a sequence of inequalities satisfying (1) or (2).

We use the sequence to consider the form of the sequents. There are several cases. As a first example, suppose that Δ contains an atom a_1 and we have a sequence $\top \leq a_1 < \ldots < a_k < \top$. Then the sequent is of the form $(\Gamma'', a_k, a_k \to a_{k-1}, \ldots, a_2 \to a_1 \Rightarrow a_1, \Delta'')$ which is easily derived using $(\to\Rightarrow)_a$ and (IDW). Suppose now that Δ contains an implication $a_1 \to a_2$ and we have a sequence $a_1 \leq a_2 < \ldots < a_k < a_1$. Then the sequent is of the form $(\Gamma'', a_1 \to a_k, \ldots, a_3 \to a_2 \Rightarrow a_1 \to a_2, \Delta'')$ which is easily derived using $(\Rightarrow\to)_a$, $(\to\Rightarrow)_a$, and (IDW). Other cases are very similar. □

Notice that unlike the completeness proof for GG, the above proof is entirely semantic. This allows us to sketch a simple proof of standard completeness for a given Hilbert system HG for Gödel logic. Suppose that $\models_{\mathrm{G}} \varphi$. Then by Theorem 5.1.4, $\vdash_{\mathrm{GG_g}} \Rightarrow \varphi$. But it is usually (depending on the Hilbert system) easy to show that $\mathrm{GG_g}$ is sound with respect to HG. So $\vdash_{\mathrm{HG}} \varphi$.

Finally, there is a way to have the best of both worlds: invertible logical rules that feature just one principal connective at a time. We allow two types of sequents, corresponding intuitively to $\leq$ and $<$, and treat sets of these sequents where exactly one formula occurs on each side. More precisely, a *sequent of relations* S is a set of ordered triples, written

$$\varphi_1 \lhd_1 \psi_1 \mid \ldots \mid \varphi_n \lhd_n \psi_n,$$

where $\varphi_i, \psi_i \in Fm_{\mathcal{L}_c}$ and $\lhd_i \in \{<, \leq\}$ for $i = 1 \ldots n$. S is G-*valid*, written $\models_{\mathrm{G}} S$, iff for all G-valuations v, $v(\varphi_i) \lhd_i v(\psi_i)$ for some $i \in \{1, \ldots, n\}$.

A sequent of relations calculus $\mathrm{GG_r}$ for Gödel logic in the language $\mathcal{L}_c$ is displayed in Figure 7, where the axioms are defined simply to be all valid atomic (containing only atoms) sequents of relations.

EXAMPLE 5.1.5. *We illustrate* $\mathrm{GG_r}$ *with the following derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{p \leq q \mid \top \leq q \mid \top \leq p \mid q < p \quad p \leq q \mid \top \leq q \mid \top \leq p \mid q \leq p}{p \leq q \mid \top \leq q \mid p \to q \leq p \mid \top \leq p}\,(\to\leq)}{p \leq q \mid \top \leq q \mid \top \leq (p \to q) \to p}\,(\leq\to)}{\top \leq p \to q \mid \top \leq (p \to q) \to p}\,(\leq\to)}{\top \leq (p \to q) \vee ((p \to q) \to p)}\,(\leq\vee)$$

Observe that the uppermost sequents of relations of this derivation are G-*valid since for any* G-*valuation* v*, always* $v(p) \leq v(q)$ *or* $v(q) < v(p)$.

Axioms: all G-valid atomic sequents of relations.

Logical rules

$$\frac{\mathcal{G} \mid \varphi \lhd \chi \mid \psi \lhd \chi}{\mathcal{G} \mid \varphi \wedge \psi \lhd \chi}\ (\wedge\lhd) \qquad \frac{\mathcal{G} \mid \chi \lhd \varphi \quad \mathcal{G} \mid \chi \lhd \psi}{\mathcal{G} \mid \chi \lhd \varphi \wedge \psi}\ (\lhd\wedge)$$

$$\frac{\mathcal{G} \mid \varphi \lhd \chi \quad \mathcal{G} \mid \psi \lhd \chi}{\mathcal{G} \mid \varphi \vee \psi \lhd \chi}\ (\vee\lhd) \qquad \frac{\mathcal{G} \mid \chi \lhd \varphi \mid \chi \lhd \psi}{\mathcal{G} \mid \chi \lhd \varphi \vee \psi}\ (\lhd\vee)$$

$$\frac{\mathcal{G} \mid \psi < \varphi \quad \mathcal{G} \mid \psi < \chi}{\mathcal{G} \mid \varphi \to \psi < \chi}\ (\to<) \qquad \frac{\mathcal{G} \mid \varphi \leq \psi \mid \chi < \psi \quad \mathcal{G} \mid \chi < \top}{\mathcal{G} \mid \chi < \varphi \to \psi}\ (<\to)$$

$$\frac{\mathcal{G} \mid \top \leq \chi \mid \psi < \varphi \quad \mathcal{G} \mid \psi \leq \chi}{\mathcal{G} \mid \varphi \to \psi \leq \chi}\ (\to\leq) \qquad \frac{\mathcal{G} \mid \varphi \leq \psi \mid \chi \leq \psi}{\mathcal{G} \mid \chi \leq \varphi \to \psi}\ (\leq\to)$$

Figure 7. The sequent of relations calculus $\mathrm{GG_r}$

The logical rules preserve G-validity in both directions, so we can reduce the G-validity of a sequent of relations to the G-validity of a set of atomic sequents of relations, and hence obtain:

THEOREM 5.1.6. $\vdash_{\mathrm{GG_r}} S$ *iff* $\models_{\mathrm{G}} S$.

Further rules may also be added, along similar lines to $\mathrm{GG_g}$, to deal directly with proving the G-validity of atomic sequents of relations.

5.2 Łukasiewicz logic and Giles's game

Just as Gödel logic G may be considered the logic of order, so Łukasiewicz logic Ł can be viewed as the logic of magnitude. In this logic, size matters. Whereas G is based on the (only) idempotent t-norm $\min$, Łukasiewicz logic is based on the nilpotent Archimedean t-norm $x * y = \max(0, x + y - 1)$. For simplicity, let us again use a restricted (but fully expressive) language, $\mathcal{L}_{Ł}$, with one binary connective $\to$ and a constant $\bot$, defining

$$\begin{array}{rcl@{\qquad}rcl} \neg\varphi & =_{\text{def}} & \varphi \to \bot & \top & =_{\text{def}} & \neg\bot \\ \varphi \,\&\, \psi & =_{\text{def}} & \neg(\varphi \to \neg\psi) & \varphi \oplus \psi & =_{\text{def}} & \neg\varphi \to \psi \\ \varphi \wedge \psi & =_{\text{def}} & \varphi \,\&\, (\varphi \to \psi) & \varphi \vee \psi & =_{\text{def}} & (\varphi \to \psi) \to \psi. \end{array}$$

An Ł-*valuation* is a function $v\colon Fm_{\mathcal{L}_Ł} \to [0,1]$ such that $v(\bot) = 0$ and

$$v(\varphi \to \psi) \quad = \quad \min(1, 1 - v(\varphi) + v(\psi))$$

where the valuations of the defined connectives emerge as expected as:

$$\begin{array}{rcl@{\qquad}rcl} v(\neg\varphi) & = & 1 - v(\varphi) & v(\top) & = & 1 \\ v(\varphi \,\&\, \psi) & = & \max(0, v(\varphi) + v(\psi) - 1) & v(\varphi \oplus \psi) & = & \min(1, v(\varphi) + v(\psi)) \\ v(\varphi \wedge \psi) & = & \min(v(\varphi), v(\psi)) & v(\varphi \vee \psi) & = & \max(v(\varphi), v(\psi)). \end{array}$$

A formula φ is Ł-*valid*, written $\models_Ł \varphi$, iff $v(\varphi) = 1$ for all Ł-valuations v.

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\text{ID}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\text{EMP}) \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \varphi}\ (\bot\Rightarrow)_{\text{Ł}}$$

Structural rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}\ (\text{EC}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}\ (\text{W})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\text{SPLIT}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\text{MIX})$$

Logical rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{\text{A}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{\text{Ł}}$$

Figure 8. The hypersequent calculus GŁ

Unlike Gödel logic, we have not as yet encountered any sequent or hypersequent calculi for Łukasiewicz logic. The most desirable solution would be just to add some further structural rules to the calculus GIMTL. However, so far no such rules have been discovered, or indeed are expected. Instead, we take a different approach: we interpret sequents outside of the language of Ł. Rather than give a direct interpretation of sequents and hypersequents as formulas, we define a semantic criterion for validity.

For a multiset of formulas Γ, we fix

$$\star_{\text{Ł}}^{v}(\Gamma) =_{\text{def}} 1 + \sum [v(\varphi) - 1 \mid \varphi \in \Gamma]$$

and define for each hypersequent $\mathcal{G}$

$$\models_{\text{Ł}} \mathcal{G} \quad \text{iff} \quad \text{for all Ł-valuations } v\text{: } \star_{\text{Ł}}^{v}(\Gamma) \leq \star_{\text{Ł}}^{v}(\Delta) \text{ for some } (\Gamma \Rightarrow \Delta) \in \mathcal{G}.$$

This definition may seem a little strange. However, notice that for a single-conclusion hypersequent $\mathcal{G}$:

$$\models_{\text{Ł}} \mathcal{G} \quad \text{iff} \quad \models_{\text{Ł}} \text{I}(\mathcal{G}).$$

In particular, $\models_{\text{Ł}} \Rightarrow \varphi$ iff $\models_{\text{Ł}} \varphi$. Moreover, as we will see shortly, there exists a quite natural interpretation of hypersequents for Łukasiewicz logic based on a simple two-player dialogue game.

The hypersequent calculus GŁ is presented in Figure 8. There are some key differences in comparison with calculi of previous sections. First, but just for convenience, the calculus is cut-free. More importantly, the logical rules for $\to$ are non-standard, the left rule has just one premise while the right rule has two. Observe, nevertheless, that the standard implication rule $(\Rightarrow\to)$ is derivable for the single-conclusion case:

$$\frac{\dfrac{\dfrac{}{\mathcal{G} \mid \Rightarrow}\ (\text{EMP})}{\mathcal{G} \mid \Gamma \Rightarrow}\ (\text{W}) \qquad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to)_{\text{Ł}}$$

Note also that the standard rules for $\wedge$ and $\vee$ can be added, while the derived initial hypersequents for $\top$ are of the form $(\mathcal{G} \mid \Gamma \Rightarrow \top)$, that is, the single-conclusion version of $(\Rightarrow \top)$. On the other hand, the appropriate rules (derived and simplified) for the defined connective $\&$ are non-standard:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \psi, \Delta \mid \Gamma \Rightarrow \bot, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \,\&\, \psi, \Delta}\ (\Rightarrow\&)_{\text{Ł}} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta}\ (\&\Rightarrow)_{\text{Ł}}$$

EXAMPLE 5.2.1. *Consider the following derivation in* GŁ*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,(\text{ID}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID})}{\psi, \varphi \Rightarrow \varphi, \psi}\,(\text{MIX})}{\psi, \psi \to \varphi \Rightarrow \varphi}\,(\to\Rightarrow)_{\text{A}} \quad \dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,(\text{ID}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\,(\text{ID})}{\psi, \varphi \Rightarrow \varphi, \psi}\,(\text{MIX})}{\psi, \psi \to \varphi, \varphi \Rightarrow \varphi, \psi}\,(\text{W})}{\psi, \psi \to \varphi \Rightarrow \varphi, \varphi \to \psi}\,(\Rightarrow\to)_{\text{Ł}}}{(\varphi \to \psi) \to \psi, \psi \to \varphi \Rightarrow \varphi}\,(\to\Rightarrow)_{\text{A}}}{(\varphi \to \psi) \to \psi \Rightarrow (\psi \to \varphi) \to \varphi}\,(\Rightarrow\to)}{\Rightarrow ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)}\,(\Rightarrow\to)$$

THEOREM 5.2.2. $\models_{\text{Ł}} \mathcal{G}$ *iff* $\vdash_{\text{GŁ}} \mathcal{G}$.

Proof. The right-to-left direction is established as usual by an induction on the height of a derivation of $\mathcal{G}$ in GŁ. For the left-to-right direction, we make use of the following GŁ-derivable rules:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{\text{Ł}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{\text{Ł}}$$

Applying these rules backwards, we get that every Ł-valid hypersequent is GŁ-derivable from Ł-valid atomic hypersequents. It therefore suffices to prove the theorem for the case where $\mathcal{G}$ is atomic. We proceed by induction on the number of distinct propositional variables occurring on the left hand side of sequents in $\mathcal{G}$. Suppose that there are none. It follows that only $\bot$ occurs on the left of sequents. We claim that there must exist a sequent where the number of occurrences of $\bot$ on the left is greater than or equal to the number of formulas on the right. If not, then defining an Ł-valuation where all variables take the value 0, we obtain a contradiction. Hence we can easily derive $\mathcal{G}$ using (EW), (W), (MIX), and $(\bot\Rightarrow)_{\text{Ł}}$.

Otherwise, we pick a variable q occurring on the left of one of the sequents of $\mathcal{G}$. If q occurs on both sides in the same sequent, then we apply (MIX) and (ID) backwards to remove it, noting that the new hypersequent is also Ł-valid. Next, we use (EC) and (SPLIT) backwards to multiply sequents, giving (for some λ) a hypersequent

$$\mathcal{G}' = (\mathcal{G}_0 \mid [\Gamma_i, [q]^\lambda \Rightarrow \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow [q]^\lambda, \Sigma_j]_{j=1}^m)$$

where q does not occur in $\mathcal{G}_0, \Gamma_i, \Delta_i, \Pi_j$, or Σ_j for $i = 1 \ldots n$ and $j = 1 \ldots m$.

Observe that $\vdash_{\text{GŁ}} \mathcal{G}$ if $\vdash_{\text{GŁ}} \mathcal{G}'$. Also $\models_{\text{Ł}} \mathcal{G}'$. Let us now define

$$\mathcal{H} = (\mathcal{G}_0 \mid [\Gamma_i, \Pi_j \Rightarrow \Sigma_j, \Delta_i]_{i=1\ldots n}^{j=1\ldots m} \mid [\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \mid [\Pi_j \Rightarrow [q]^\lambda, \Sigma_j]_{j=1}^m).$$

Clearly $\mathcal{H}$ contains fewer distinct variables occurring on the left of sequents. Also $\mathcal{G}'$ is derivable from $\mathcal{H}$. Reasoning backwards, we apply (EC) and (SPLIT) to $\mathcal{G}'$ to combine sequents of the form $(\Gamma_i, [q]^\lambda \Rightarrow \Delta_i)$ and $(\Pi_j \Rightarrow [q]^\lambda, \Sigma_j)$ into one:

Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\text{ID}) \qquad \Rightarrow\ (\text{EMP}) \qquad \frac{}{\Gamma, \bot \Rightarrow \varphi}\ (\bot\Rightarrow)_{Ł}$$

Structural rules

$$\frac{\Gamma^n \Rightarrow \Delta^n}{\Gamma \Rightarrow \Delta}\ (\text{SC}_n)\ n \geq 2 \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{W}) \qquad \frac{\Gamma_1 \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\text{MIX})$$

Logical rules

$$\frac{\Gamma, \psi, \psi \to \varphi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)^s_{Ł} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{Ł}$$

Figure 9. The sequent calculus GŁ_s

$(\Gamma_i, \Pi_j, [q]^\lambda \Rightarrow [q]^\lambda, \Delta_i, \Sigma_j)$. Then we apply (MIX) and (ID) backwards to remove the balanced occurrences of q, and (W) backwards to $(\Gamma_i, [q]^\lambda \Rightarrow \Delta_i)$ to get $(\Gamma_i \Rightarrow \Delta_i)$. Hence it is sufficient to show that $\mathcal{H}$ is Ł-valid, since then by the induction hypothesis $\vdash_{\text{GŁ}} \mathcal{H}$. Suppose otherwise for a contradiction, i.e., that there exists an Ł-valuation v such that $\star^v_{Ł}(\Gamma) > \star^v_{Ł}(\Delta)$ for all $(\Gamma \Rightarrow \Delta) \in \mathcal{H}$. We define

$$\begin{aligned} x &= \max(\{\star^v_{Ł}(\Delta_i) - \star^v_{Ł}(\Gamma_i) \mid 1 \leq i \leq n\} \cup \{-\lambda\}) \\ y &= \min(\{\star^v_{Ł}(\Pi_j) - \star^v_{Ł}(\Sigma_j) \mid 1 \leq j \leq m\} \cup \{0\}). \end{aligned}$$

Notice first that $x < y$. Otherwise, $\star^v_{Ł}(\Gamma_i) + \star^v_{Ł}(\Pi_j) \leq \star^v_{Ł}(\Sigma_j) + \star^v_{Ł}(\Delta_i)$ for some i, j: a contradiction. Now change v so that $x < \lambda(v(q) - 1) < y$, noting that $v(q) \in (0, 1)$. Then for $i = 1 \ldots n$ and $j = 1 \ldots m$:

$$\star^v_{Ł}(\Delta_i) - \star^v_{Ł}(\Gamma_i) < \lambda(v(q) - 1) \quad \text{and} \quad \lambda(v(q) - 1) < \star^v_{Ł}(\Pi_j) - \star^v_{Ł}(\Sigma_j).$$

Hence, rearranging, $\star^v_{Ł}(\Gamma_i \uplus [q]^\lambda) > \star^v_{Ł}(\Delta_i)$ and $\star^v_{Ł}(\Pi_j) > \star^v_{Ł}(\Sigma_j \uplus [q]^\lambda)$. But then $\not\models_{Ł} \mathcal{G}'$, a contradiction. □

A nice by-product of this completeness proof is a syntactic proof of the decidability of validity in Łukasiewicz logic. A hypersequent is Ł-valid iff it is derivable by applying the rules $(\to\Rightarrow)_{Ł}$ and $(\Rightarrow\to)_{Ł}$ exhaustively and then iteratively removing occurrences of each variable as described above. Hence we have a terminating procedure to decide whether any given formula φ is or is not Ł-valid.

Defining a hypersequent calculus for Ł requires ingenuity, but it does not come as a complete surprise. The existence of a sequent calculus for Ł on the other hand is quite unexpected. The key idea here is to represent differences between sequents occurring in a hypersequent using implicational formulas. Consider again the rule $(\to\Rightarrow)_{Ł}$. Using the fact that $v(\varphi) + v(\varphi \to \psi) = v(\psi) + v(\psi \to \varphi)$ for all Ł-valuations v, we replace this rule at the sequent level with

$$\frac{\Gamma, \psi, \psi \to \varphi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)^s_{Ł}$$

Notice that the premise $(\Gamma, \psi, \psi \to \varphi \Rightarrow \varphi, \Delta)$ is derivable both from $(\Gamma, \psi \Rightarrow \varphi, \Delta)$ using (W), and from $(\Gamma \Rightarrow \Delta)$ using $(\to\Rightarrow)^s_{Ł}$ again together with the sequent rules (MIX), (ID), and (W). Intuitively, we think of $(\Gamma, \psi, \psi \to \varphi \Rightarrow \varphi, \Delta)$ as representing both sequents. The combination of such sequents, performed by the split rule in the hypersequent calculus, is achieved here by the sequent contraction rules $(\mathrm{SC_n})$ for $n \geq 2$. The resulting calculus $\mathrm{GŁ_s}$ is displayed in Figure 9.

EXAMPLE 5.2.3. *Compare the following* $\mathrm{GŁ_s}$*-derivation with the* $\mathrm{GŁ}$*-derivation of the same sequent given in Example 5.2.1, recalling that the standard single-conclusion implication right rule* $(\Rightarrow\to)$ *is derivable using* $(\Rightarrow\to)_{Ł}$*,* (W)*, and* (EMP)*:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{ID}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,(\mathrm{ID}) \quad \dfrac{}{\varphi \to \psi \Rightarrow \varphi \to \psi}\,(\mathrm{ID})}{\psi, \varphi \to \psi \Rightarrow \psi, \varphi \to \psi}\,(\mathrm{MIX})}{\psi \to (\varphi \to \psi), \psi, \varphi \to \psi \Rightarrow \psi, \varphi \to \psi}\,(\mathrm{W})}{(\varphi \to \psi) \to \psi, \varphi \to \psi \Rightarrow \psi}\,(\to\Rightarrow)^s_{Ł}}{(\varphi \to \psi) \to \psi, \varphi \to \psi, \varphi \Rightarrow \varphi, \psi}\,(\mathrm{MIX})}{(\varphi \to \psi) \to \psi, \psi \to \varphi \Rightarrow \varphi}\,(\to\Rightarrow)^s_{Ł}}{(\varphi \to \psi) \to \psi \Rightarrow (\psi \to \varphi) \to \varphi}\,(\Rightarrow\to)}{\Rightarrow ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)}\,(\Rightarrow\to)
$$

The completeness proof for $\mathrm{GŁ_s}$ relies on a (quite complicated) translation of sequent derivations into derivations in the hypersequent calculus $\mathrm{GŁ}$, and is omitted here.

THEOREM 5.2.4. $\models_{Ł} S$ *iff* $\vdash_{\mathrm{GŁ_s}} S$.

We conclude our discussion of Łukasiewicz logic with a dialogue game introduced by Robin Giles in the 1970s (for details see the historical remarks at the end of the chapter) that has an intriguing connection to the hypersequent calculus described above. Suppose that two players — me and you — agree to pay 1$ to their opponent for every false statement that they make. An *elementary state* of the game consists of a multiset of atoms $[a_1, \ldots, a_m]$ asserted by you and a multiset of atoms $[b_1, \ldots, b_n]$ asserted by me, denoted $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$. Each atom a is read as "the (repeatable) *elementary (yes/no) experiment* E_a yields a positive result". If a is either always true or false, the setting is classical. The situation is more interesting, however, if the same experiment can yield different results when repeated. In this case, for every run of the game, a fixed *risk value* $\langle q \rangle \in [0, 1]$ is associated with each variable q, where $\langle \bot \rangle = 1$. The *risk* associated with a multiset of atoms is then $\langle [a_1, \ldots, a_m] \rangle =_{\text{def}} \langle a_1 \rangle + \ldots + \langle a_m \rangle$. That is, my risk corresponds to the amount of money that I expect to have to pay to you according to the results of the experiments associated with the atoms that I assert. Hence, for an elementary state $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$, the condition $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$ expresses that I do not expect any loss (but possibly some gain).

More generally, a *dialogue state* (*d-state*) of the game is denoted $[\Gamma \,\|\, \Delta]$ where Γ and Δ are finite multisets of formulas currently asserted by you and me, respectively. Non-atomic formulas in Γ and Δ may then be decomposed via the following dialogue rule for implication:

> $(R_{\to})$ If I assert $\varphi \to \psi$, then whenever you attack this statement by asserting φ, I must assert also ψ. (And *vice versa*, i.e., for the roles of me and you switched.)

This rule reflects the idea that implication is characterized by the principle that asserting "if φ, then ψ" obliges the assertion also of ψ if the opponent in a dialogue asserts φ. However, a player may also choose never to attack the opponent's assertion of $\varphi \to \psi$. A *round* with *initiator* P (me or you) and *respondent* Q (the other player) is therefore a transition from one d-state to another *successor d-state* that consists of two *moves*:

1. P picks one of the formulas $\varphi \to \psi$ asserted by Q.
2. P either *attacks* this formula by asserting φ or *grants* the formula, meaning that P declares never to attack that particular occurrence. In the first case, Q must respond immediately by asserting ψ; in the second, no action is required from Q.

Each formula can be attacked or granted at most once. Hence the formula chosen by the initiator is removed from the d-state. In denoting strategies as trees, below, we use *intermediary states* or, for short, *i-states* to reflect the initiator's choice of the formula that gets attacked or granted. Intermediary states are exactly the same as the preceding d-state, except that the formula chosen by the initiator is marked (denoted by underlining).

To define games precisely, we also need to know whose turn it is to initiate the next round for each non-elementary d-state. Formally, we define a *regulation* as a function ρ that assigns to each non-elementary d-state either the label $\mathbf{Y}$, for "*you initiate the next round*", or the label $\mathbf{I}$, for "*I initiate the next round*", indicating that the possible runs of the game are constrained accordingly. A regulation is *consistent* if the label $\mathbf{Y}$ (or $\mathbf{I}$) is only assigned to d-states where such an initiating move is possible, i.e., where there is a formula $\varphi \to \psi$ among my (or your) currently asserted formulas. The correct label for an i-state is determined by the label of the immediately preceding d-state.

A *game form* $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a tree of states with the initial d-state $[\Gamma \parallel \Delta]$ as root, where the successor nodes to any state S are the states that result from legal moves at S according to the consistent regulation ρ. In particular, the leaf nodes of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ are the reachable elementary states. A *game* consists of a game form $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ together with a risk assignment $\langle \cdot \rangle$, and a *run* of the game is a branch of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$. In other words, a run consists of a sequence of alternating d-states and i-states, obtained from successive moves as described above, beginning with the initial d-state $[\Gamma \parallel \Delta]$ and ending in an elementary state.

EXAMPLE 5.2.5. *Consider the d-state $[p \to q \parallel a \to b, c \to d]$. If it is my turn to initiate the next round (indicated by $^{\mathbf{I}}$), then I can only either attack or grant your statement $p \to q$. Accordingly, there are two ways that runs of the game can continue:*

$$
\begin{array}{ccc}
[p \to q \parallel a \to b, c \to d]^{\mathbf{I}} & \qquad & [p \to q \parallel a \to b, c \to d]^{\mathbf{I}} \\
| & & | \\
[\underline{p \to q} \parallel a \to b, c \to d]^{\mathbf{I}} & & [\underline{p \to q} \parallel a \to b, c \to d]^{\mathbf{I}} \\
| & & | \\
[q \parallel p, a \to b, c \to d] & & [\,\parallel a \to b, c \to d].
\end{array}
$$

If it is your turn to move, then there are two successor i-states and hence four possible continuations depending on (1) which of my two statements you choose, and (2) whether you decide to attack or grant the chosen formula:

$$
\begin{array}{cc}
[p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} & [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\
| & | \\
[p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} & [p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} \\
| & | \\
[p \to q, a \,\|\, b, c \to d] & [p \to q \,\|\, c \to d]
\end{array}
$$

$$
\begin{array}{cc}
[p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} & [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\
| & | \\
[p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} & [p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} \\
| & | \\
[p \to q, c \,\|\, a \to b, d] & [p \to q \,\|\, a \to b].
\end{array}
$$

Suppose that a run of $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ with risk assignment $\langle \cdot \rangle$ ends with the elementary state $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$. We say that I *win* in that run if I do not expect any loss of money resulting from betting on results of the corresponding elementary experiments: more formally, I win if $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$. In general, a game *strategy* for a particular player is a function from states to states that determines every choice of a legal move by that player but leaves all choices of the other player open. In our context this means that a strategy for me is obtained from a game form by (iteratively from the root) deleting all but one successor of every state labelled **I**. A strategy for a game form $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ is called a *winning strategy (for me) for a risk assignment* $\langle \cdot \rangle$ if $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$ holds for each of its leaf nodes $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$.

The following result expresses the key relationship between Ł-validity and winning strategies in the game (see the historical remarks for references):

THEOREM 5.2.6. *A formula φ is Ł-valid iff I have a winning strategy for the game $\mathbf{G}([\,\|\, \varphi], \rho)$ with any risk assignment $\langle \cdot \rangle$, where ρ is an arbitrary consistent regulation.*

To connect Giles's game to cut-free proofs in an appropriate calculus for Łukasiewicz logic, we must abstract from particular risk assignments and look rather at *disjunctive (winning) strategies* that arise when *disjunctions of states* instead of single game states are considered. Let us use $D = S_1 \bigvee \ldots \bigvee S_n$ to denote a *state disjunction*. Since the order of its *component states* (ordinary game states) $S_1, \ldots, S_n$ is irrelevant, a disjunctive state may be viewed as a multiset of states. A *disjunctive strategy* for D respecting a regulation ρ is a tree of state disjunctions with root D where the successor nodes are in principle determined in the same way as for ordinary strategies. However, we also allow for the possibility of *duplicating* a component of a state in order to let disjunctive strategies for a player P record different ways of proceeding for P in identical component states. More precisely, there are two kinds of non-leaf nodes $D = S_1 \bigvee \ldots \bigvee S_n$ in a disjunctive strategy for P:

1. *Playing nodes*, focused on some component S_i of D. The successor nodes are like those for S_i in ordinary strategies, except for the presence of additional components (that remain unchanged). I.e., if, according to ρ, it is P's turn to play at S_i, then there is a single successor state disjunction where the component S_i of D is replaced by some S_i' corresponding to some move of P. If the opposing player Q is to move at S_i, then all possible moves of Q determine the successor nodes of D where S_i is replaced by a state obtained by the corresponding move.

2. *Duplicating nodes*, where the single successor node is obtained by duplicating one of the components in D.

A *disjunctive winning strategy (for me)* for the family of instances of Giles's game based on $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ is a disjunctive strategy where in each leaf node for *every* risk assignment $\langle\cdot\rangle$ there is at least one component, i.e., elementary d-state, $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$ such that $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$.

As should be fairly obvious, d-states may be understood as sequents, and state disjunctions as hypersequents. Furthermore, a cursory analysis of disjunctive strategies reveals that (ignoring marking of formulas and the different notation) these are derivations of hypersequents from atomic hypersequents using (EC), (EW), $(\Rightarrow\to)_Ł$, the derived rule $(\to\Rightarrow)_Ł$ used in the proof of Theorem 5.2.2, and the "redundant rule" (representing the introduction of intermediate states)

$$\frac{\mathcal{G} \ldots \mathcal{G}}{\mathcal{G}} \ (\text{R})$$

Moreover, if a disjunctive strategy is winning, then the atomic hypersequents corresponding to the leaf nodes are Ł-valid. It is not the case that any derivation from Ł-valid hypersequents using (EC), (EW), $(\to\Rightarrow)_Ł$, $(\Rightarrow\to)_Ł$, and (R) corresponds directly to a disjunctive winning strategy, but given such a derivation, a disjunctive winning strategy can be constructed for any consistent regulation.

5.3 Product logic and related systems

Product logic P, the third fundamental fuzzy logic, possesses features of both of its more famous siblings. Roughly speaking, it behaves like Ł on the interval $(0, 1]$, and like G at 0. Let us work again with a more restricted language $\mathcal{L}_b$ with binary connectives $\&$ and $\to$, and a constant $\bot$, defining

$$\begin{array}{lll@{\qquad}lll} \neg\varphi & =_{\text{def}} & \varphi \to \bot & \varphi \wedge \psi & =_{\text{def}} & \varphi \,\&\, (\varphi \to \psi) \\ \top & =_{\text{def}} & \neg\bot & \varphi \vee \psi & =_{\text{def}} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi). \end{array}$$

P-*valuations* are then functions $v\colon Fm_{\mathcal{L}_b} \to [0, 1]$ such that $v(\bot) = 0$ and

$$v(\varphi \,\&\, \psi) = v(\varphi) \cdot v(\psi) \qquad v(\varphi \to \psi) = \begin{cases} v(\psi)/v(\varphi) & \text{if } v(\varphi) > v(\psi) \\ 1 & \text{otherwise.} \end{cases}$$

A formula φ is P-*valid*, written $\models_{\mathrm{P}} \varphi$, iff $v(\varphi) = 1$ for all P-valuations v.

In developing Gentzen systems for P, the problems are similar to those encountered for Ł. It does not seem to be possible to obtain a calculus by adding structural rules to a standard calculus like GMTL. Rather, as for Ł, we make use of a non-standard interpretation for sequents – this time within the language of the logic – and then develop tailored logical rules for that interpretation.

$$\begin{array}{rcl} \mathrm{I_P}(\Gamma \Rightarrow \Delta) & =_{\text{def}} & \&\Gamma \to \&\Delta \\ \mathrm{I_P}(S_1 \mid \ldots \mid S_n) & =_{\text{def}} & \mathrm{I_P}(S_1) \vee \ldots \vee \mathrm{I_P}(S_n). \end{array}$$

Hence, letting $\star_{\mathrm{P}}^v(\Gamma) =_{\text{def}} \prod_{\varphi \in \Gamma} v(\varphi)$ where $\star_{\mathrm{P}}^v([]) = 1$, we have

$$\models_{\mathrm{P}} \mathrm{I_P}(\mathcal{G}) \quad \text{iff} \quad \text{for all valuations } v,\ \star_{\mathrm{P}}^v(\Gamma) \leq \star_{\mathrm{P}}^v(\Delta) \ \text{ for some } \ (\Gamma \Rightarrow \Delta) \in \mathcal{G}.$$

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\textsc{emp}) \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow \Delta}\ (\bot\Rightarrow)$$

Structural rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}\ (\textsc{ew}) \qquad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}\ (\textsc{ec}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta}\ (\textsc{w})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{mix})$$

Logical rules

$$\frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta}\ (\&\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \,\&\, \psi, \Delta}\ (\Rightarrow\&)_{\mathrm{A}}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta}\ (\neg\Rightarrow)_{\mathrm{P}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{\text{Ł}}$$

$$\frac{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{\mathrm{P}}$$

Figure 10. The hypersequent calculus GP

The hypersequent calculus GP based on this interpretation is displayed in Figure 10, where $\neg$ is used here simply as a convenient abbreviation for $\varphi \to \bot$.

EXAMPLE 5.3.1. *We illustrate the calculus with the following derivation, noting that the single-conclusion rule* $(\Rightarrow\to)$ *is derivable in* GP *exactly as in* GŁ*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\textsc{id}) \quad \dfrac{}{\psi \Rightarrow \psi}\,(\textsc{id})}{\varphi, \psi \Rightarrow \varphi, \psi}\,(\textsc{mix})}{\varphi \,\&\, \psi \Rightarrow \varphi, \psi}\,(\&\Rightarrow)}{\neg\neg\varphi, \varphi \,\&\, \psi \Rightarrow \varphi, \psi}\,(\textsc{w}) \qquad \dfrac{\dfrac{}{\neg\varphi \Rightarrow \neg\varphi}\,(\textsc{id})}{\neg\neg\varphi, \neg\varphi \Rightarrow \psi}\,(\neg\Rightarrow)_{\mathrm{P}}}{\dfrac{\neg\neg\varphi, \varphi \to (\varphi \,\&\, \psi) \Rightarrow \psi}{\neg\neg\varphi \Rightarrow (\varphi \to (\varphi \,\&\, \psi)) \to \psi}\,(\Rightarrow\to)}\,(\to\Rightarrow)_{\mathrm{P}}}{\Rightarrow \neg\neg\varphi \to ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)}\,(\Rightarrow\to)$$

Soundness is proved in the usual way. For completeness, we proceed semantically. First note that the following rule is GP-derivable using (EC), (W), $(\to\Rightarrow)_{\mathrm{P}}$, and (EW):

$$\frac{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)^{i}_{\mathrm{P}}$$

Now let GP^{+} be GP extended with the following sound and invertible (with respect to P-validity) "decomposition rules" for negated formulas:

$$\frac{\mathcal{G} \mid \Gamma, \neg\psi \Rightarrow \Delta \mid \varphi \Rightarrow \psi}{\mathcal{G} \mid \Gamma, \neg(\varphi \to \psi) \Rightarrow \Delta}\ (\neg\to\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \neg\psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \neg(\varphi \,\&\, \psi) \Rightarrow \Delta}\ (\neg\&\Rightarrow)$$

Completeness for GP^{+} is shown by first using invertible logical rules and the extra rules for negation to reduce the P-validity of hypersequents to the P-validity of hypersequents containing atoms and possibly also negated variables on the left of sequents, and then showing that such hypersequents are derivable using a similar argument to that employed for GŁ in Theorem 5.2.2. Finally, completeness for GP is obtained by showing (inductively) that applications of the extra rules can be eliminated from derivations in GP^+.

THEOREM 5.3.2. $\models_{\text{P}} \text{I}_{\text{P}}(\mathcal{G})$ *iff* $\vdash_{\text{GP}} \mathcal{G}$.

Similarly to the case of Gödel logic, GP can be used to establish standard completeness for a given axiomatization HP of product logic. If $\models_{\text{P}} \varphi$, then by the completeness of GP, $\vdash_{\text{GP}} \Rightarrow \varphi$. But it is usually (depending on the Hilbert system) straightforward to show that GP is sound with respect to HP. Hence $\vdash_{\text{HP}} \varphi$. Moreover, following the same ideas as for Łukasiewicz logic, decidability of validity in product logic can be deduced from the completeness proof for GP^+ and a sequent calculus GP_{s} can be defined.

Some related fuzzy logics, not as important as the three just covered, but interesting nonetheless, can be tackled using similar techniques. Recall that *cancellative hoop logic* CHL, the logic of cancellative hoops, emerges by removing $\bot$ from the language of P and restricting P-valuations to the half-open interval $(0, 1]$. A hypersequent calculus for this logic is obtained by removing the rule $(\bot\Rightarrow)_{\text{Ł}}$ from GŁ and adding the rules for $\&$ of GP. Further examples are *cross ratio logic* CRL, the logic based on the cross-ratio uninorm, and *abelian logic* A, the logic of abelian lattice-ordered groups.

5.4 Uniform systems

The calculi for the fundamental logics defined above are nice in many respects, but they do seem rather arbitrary. For each logic we have simply picked rules that work. In contrast, the systems of the previous sections have the same logical rules and differ only at the structural level. Here we define a similar framework for Ł, G, and P. That is, we define uniform logical rules and distinguish the logics using only structural rules. The price we pay is more structure: hypersequents are generalized to "relational hypersequents" with two kinds of sequents.

We will use a language $\mathcal{L}_f$ with binary connectives $\to, \&, \wedge, \vee$ and constants $\bot, \top$ that is adequate for Ł, G, and P, and indeed any logic based on a residuated t-norm. A *relational hypersequent (r-hypersequent)* is a finite multiset of ordered triples, written

$$\Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n$$

where $\lhd_i \in \{<, \leq\}$ and Γ_i and Δ_i are finite multisets of formulas for $i = 1 \ldots n$. Note that a hypersequent can be treated as an r-hypersequent with just one relation symbol, and that a sequent of relations can be treated as an r-hypersequent where all multisets Γ_i, Δ_i contain exactly one formula.

Validity for r-hypersequents is defined for each logic individually, understanding $\mid$ as before as a meta-level disjunction, where $<$ and $\leq$ denote inequalities between combinations (different for each logic) of truth values of formulas. The symbols $<$ and $\leq$ therefore have two uses: a syntactic one as part of an r-hypersequent, and a semantic one as an inequality holding between mathematical expressions. Often we will use $\lhd$ to stand uniformly for $\leq$ or $<$ (in either sense).

$$\frac{\mathcal{G} \mid \Gamma, \psi \lhd \varphi, \Delta \mid \varphi \leq \psi \quad \mathcal{G} \mid \Gamma \lhd \Delta \mid \psi < \varphi}{\mathcal{G} \mid \Gamma, \varphi \to \psi \lhd \Delta} \ (\to\lhd) \qquad \frac{\mathcal{G} \mid \Gamma \lhd \Delta \quad \mathcal{G} \mid \Gamma, \varphi \lhd \psi, \Delta \mid \varphi \leq \psi}{\mathcal{G} \mid \Gamma \lhd \varphi \to \psi, \Delta} \ (\lhd\to)$$

$$\frac{\mathcal{G} \mid \Gamma \lhd \bot, \Delta \mid \Gamma \lhd \varphi, \psi, \Delta}{\mathcal{G} \mid \Gamma \lhd \varphi \,\&\, \psi, \Delta} \ (\lhd\&) \qquad \frac{\mathcal{G} \mid \Gamma, \varphi, \psi \lhd \Delta \quad \mathcal{G} \mid \Gamma, \bot \lhd \Delta}{\mathcal{G} \mid \Gamma, \varphi \,\&\, \psi \lhd \Delta} \ (\&\lhd)$$

$$\frac{\mathcal{G} \mid \Gamma \lhd \varphi, \Delta \quad \mathcal{G} \mid \Gamma \lhd \psi, \Delta}{\mathcal{G} \mid \Gamma \lhd \varphi \wedge \psi, \Delta} \ (\lhd\wedge) \qquad \frac{\mathcal{G} \mid \Gamma, \varphi \lhd \Delta \mid \Gamma, \psi \lhd \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \lhd \Delta} \ (\wedge\lhd)$$

$$\frac{\mathcal{G} \mid \Gamma \lhd \varphi, \Delta \mid \Gamma \lhd \psi, \Delta}{\mathcal{G} \mid \Gamma \lhd \varphi \vee \psi, \Delta} \ (\lhd\vee) \qquad \frac{\mathcal{G} \mid \Gamma, \varphi \lhd \Delta \quad \mathcal{G} \mid \Gamma, \psi \lhd \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \lhd \Delta} \ (\vee\lhd)$$

Figure 11. Uniform relational hypersequent rules

For $\mathrm{L} \in \{\text{Ł}, \mathrm{G}, \mathrm{P}\}$, we define $\star_{\mathrm{L}}^{v}([]) = 1$ and for $\Gamma \neq []$:

$$\begin{array}{rcl} \star_{\text{Ł}}^{v}(\Gamma) & = & 1 + \sum[v(\varphi) - 1 \mid \varphi \in \Gamma] \\ \star_{\mathrm{G}}^{v}(\Gamma) & = & \min[v(\varphi) \mid \varphi \in \Gamma] \\ \star_{\mathrm{P}}^{v}(\Gamma) & = & \prod[v(\varphi) \mid \varphi \in \Gamma]. \end{array}$$

An r-hypersequent $\mathcal{G}$ is said to be L-*valid*, written $\models_{\mathrm{L}} \mathcal{G}$, iff for each L-valuation v, $\star_{\mathrm{L}}^{v}(\Gamma) \lhd \star_{\mathrm{L}}^{v}(\Delta)$ for some $(\Gamma \lhd \Delta) \in \mathcal{G}$.

Notice immediately that for any formula φ and $\mathrm{L} \in \{\text{Ł}, \mathrm{G}, \mathrm{P}\}$:

$$\models_{\mathrm{L}} \ \leq \varphi \qquad \text{iff} \qquad \models_{\mathrm{L}} \varphi.$$

Hence we can express that a single formula is L-valid, as well as other relationships such as $\models_{\mathrm{L}} \varphi < \psi$ that cannot be expressed using the L-validity of a formula.

EXAMPLE 5.4.1. $\mathcal{G} = (r \leq r, q \mid p, q < p)$ *is* Ł*-valid since for any* Ł*-valuation* v*:*

$$\begin{array}{lcl} v(q) = 1 & \Rightarrow & \star_{\text{Ł}}^{v}[r] = v(r) = 1 + (v(r) - 1) + (v(q) - 1) = \star_{\text{Ł}}^{v}[r, q] \\ v(q) < 1 & \Rightarrow & \star_{\text{Ł}}^{v}[p, q] = 1 + (v(p) - 1) + (v(q) - 1) < v(p) = \star_{\text{Ł}}^{v}[p]. \end{array}$$

However, $\mathcal{G}$ *is not* L*-valid for* $\mathrm{L} \in \{\mathrm{G}, \mathrm{P}\}$*, since if* $v(p) = v(q) = 0$ *and* $v(r) > 0$*, then*

$$\star_{\mathrm{L}}^{v}[r] = v(r) > 0 = \star_{\mathrm{L}}^{v}[r, q] \quad \textit{and} \quad \star_{\mathrm{L}}^{v}[p, q] = 0 = v(p) = \star_{\mathrm{L}}^{v}[p].$$

Our reward for this greater flexibility, both in the structures and their interpretations, is the set of uniform rules displayed in Figure 11 (recalling that $\lhd$ is uniformly either $\leq$ or $<$ in each instance of a rule). Notice that the rules for $\to$, $\wedge$, and $\vee$ have the subformula property, but the rules for & do not ($\bot$ appears in the premises and possibly not the conclusion). This is another cost of uniformity. In the cases of G and P, we could remove the right premise of ($\&\lhd$), and $(\Gamma \lhd \bot, \Delta)$ in the premise of ($\lhd\&$), while for Ł we could make do with just rules for $\to$.

The uniform rules are sound and invertible for Ł, G, and P (i.e., for $\mathrm{L} \in \{\text{Ł}, \mathrm{G}, \mathrm{P}\}$, the conclusion of any rule instance is L-valid iff all the premises are L-valid). Moreover, it is easy to see that for each rule instance, the multiset complexity of the premises is strictly less than the multiset complexity of the conclusion. Hence a sound and complete calculus for L where $\mathrm{L} \in \{\mathrm{Ł}, \mathrm{G}, \mathrm{P}\}$ consists of the uniform rules extended with (as initial r-hypersequents) all L-valid atomic r-hypersequents. Indeed, it can be shown that the sets of L-valid atomic r-hypersequents Ł, G, and P are polynomial time, and by revising the uniform rules slightly, the systems can then be used to establish co-NP completeness for the sets of valid formulas for these logics. Alternatively, structural rules – different for each logic – can be given for dealing with atomic r-hypersequents. Details may be found in the references given at the end of this chapter.

6 Quantifiers and modalities

In this section, we consider proof-theoretic methods for three extensions of fuzzy logics beyond the propositional level: first-order quantifiers, propositional quantifiers, and modalities.

6.1 First-order quantifiers

Surprisingly perhaps, many of the methods and results for propositional logics transfer unscathed to the first-order level. We can extend the hypersequent calculi described in Sections 3 and 4 with rules for the universal quantifier $\forall$ and existential quantifier $\exists$, and obtain completeness with respect to appropriate Hilbert systems or classes of algebras. Moreover, although the logics are undecidable, analogues of Herbrand's theorem and Skolemization – twin pillars of theorem proving in classical logic – can be established for their prenex fragments. In the case of the non recursively axiomatizable first-order Łukasiewicz logic, the results are necessarily weaker; however, an alternative "approximate" Herbrand theorem can be obtained, leading in turn to a cut-free hypersequent calculus with an infinitary rule.

Let us suppose that we have already defined a hypersequent calculus GL over the usual propositional language $\mathcal{L}_p$ and that $\mathbb{L}$ is a fixed countable first-order language with function symbols, relation symbols, and variables. To simplify matters, we make a syntactic distinction between bound variables, denoted by x, y and free variables, denoted by a, b, and assume that only bound variables are quantified and only free variables occur freely. Terms are denoted by t, and first-order formulas (as in the propositional case) by φ, ψ, χ, writing $\varphi(\bar{a})$ to denote that the free variables of φ are among those in $\bar{a} = a_1, \ldots, a_n$ and $\varphi(\bar{t})$ to denote distinguished occurrences of terms $\bar{t} = t_1, \ldots, t_n$ in φ. We then define GL$\forall$ to consist of all instances of rule schema of GL with $\mathbb{L}$-formulas together with the following quantifier rules:

$$\frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta}\ (\forall\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta}\ (\Rightarrow\forall)$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}\ (\exists\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta}\ (\Rightarrow\exists)$$

where a does not occur in the lower hypersequent in $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$.

Note that unlike in the usual quantifier axioms, here we do not require that the term t is substitutable for x in $(\forall\Rightarrow)$ and $(\Rightarrow\exists)$: no free variable in t can be bound in $\varphi(t)$ since free variables and bound variables are distinguished.

EXAMPLE 6.1.1. *We can use these quantifier rules and the core rules for $\to$ to give sequent derivations of standard existential quantifier axioms (noting that, by definition, x does not occur freely in ψ):*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi(a)\Rightarrow\varphi(a)}\,(\text{ID}) \qquad \dfrac{}{\psi\Rightarrow\psi}\,(\text{ID})}{\varphi(a)\to\psi,\varphi(a)\Rightarrow\psi}\,(\to\Rightarrow)}{(\forall x)(\varphi\to\psi),\varphi(a)\Rightarrow\psi}\,(\forall\Rightarrow)}{(\forall x)(\varphi\to\psi),(\exists x)\varphi\Rightarrow\psi}\,(\exists\Rightarrow)}{(\forall x)(\varphi\to\psi)\Rightarrow(\exists x)\varphi\to\psi}\,(\Rightarrow\to)}{\Rightarrow(\forall x)(\varphi\to\psi)\to((\exists x)\varphi\to\psi)}\,(\Rightarrow\to)$$

We can also derive the so-called "shifting law of quantifiers" axioms. Recall that $(\Rightarrow\vee)$ and $(\wedge\Rightarrow)$ are derived rules of GUL, *and, from Example 3.2.2, that $\vdash_{\mathrm{GUL}} \varphi\vee\psi\Rightarrow\varphi \mid \varphi\vee\psi\Rightarrow\psi$. Then we have a derivation in* GUL$\forall$ *(noting again that, by definition, x does not occur freely in φ):*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi\vee\psi(a)\Rightarrow\varphi \mid \varphi\vee\psi(a)\Rightarrow\psi(a)}{\varphi\vee\psi(a)\Rightarrow\varphi \mid (\forall x)(\varphi\vee\psi)\Rightarrow\psi(a)}\,(\forall\Rightarrow)}{(\forall x)(\varphi\vee\psi)\Rightarrow\varphi \mid (\forall x)(\varphi\vee\psi)\Rightarrow\psi(a)}\,(\forall\Rightarrow)}{(\forall x)(\varphi\vee\psi)\Rightarrow\varphi \mid (\forall x)(\varphi\vee\psi)\Rightarrow(\forall x)\psi}\,(\Rightarrow\forall)}{(\forall x)(\varphi\vee\psi)\Rightarrow\varphi\vee(\forall x)\psi}\,(\Rightarrow\vee)}{\Rightarrow(\forall x)(\varphi\vee\psi)\to(\varphi\vee(\forall x)\psi)}\,(\Rightarrow\to)$$

Such axioms are derivable in calculi for first-order classical logic, but not in calculi for first-order intuitionistic logic or many other first-order substructural logics.

Soundness and completeness are established for first-order Gentzen systems with respect to Hilbert systems similarly to the propositional case by showing that the new quantifier rules preserve derivability in the Hilbert system, and that the extra axioms are derivable and the new rules preserve derivability in the hypersequent calculus. Cut elimination also follows the same pattern as the propositional case. To take care of substituting variables, however, we denote a hypersequent containing a distinguished free variable a by $\mathcal{G}(a)$, and make use of the following lemma, proved by a straightforward induction on the height of a derivation.

LEMMA 6.1.2. *Let* GL *be an extension of* GUL *with substitutive single-conclusion rules or of* GIUL *with substitutive rules. If $d;\{\mathcal{G}_1(a),\ldots,\mathcal{G}_n(a)\}\vdash_{\mathrm{GL}\forall}\mathcal{G}(a)$ and t is a term with variables not occurring in d, then $d';\{\mathcal{G}_1(t),\ldots,\mathcal{G}_n(t)\}\vdash_{\mathrm{GL}\forall}\mathcal{G}(t)$ for some derivation d' with $\mathrm{h}(d')=\mathrm{h}(d)$.*

THEOREM 6.1.3. *Let* GL *be an extension of* GUL *with substitutive single-conclusion rules or of* GIUL *with substitutive rules. Then cut elimination holds for* GL$\forall$.

Proof. As in Theorem 4.1.1, it is sufficient to prove that for any hypersequent $\mathcal{G}$ and hypersequent $\mathcal{H}$ with marked formula φ:

$$\text{If } d_{\mathcal{G}} \vdash_{\mathrm{GL}\forall^\circ} \mathcal{G} \text{ and } d_{\mathcal{H}} \vdash_{\mathrm{GL}\forall^\circ} \mathcal{H}, \text{ then } \vdash_{\mathrm{GL}\forall^\circ} \mathcal{G}' \text{ for all } \mathcal{G}' \in \mathrm{CUT}(\mathcal{G}, \mathcal{H}).$$

We prove the claim as before by induction on the lexicographically ordered triple

$$\langle \mathrm{cp}(\varphi), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d_{\mathcal{G}}) \rangle$$

assuming, using Lemma 6.1.2, that any new free variables introduced (upwards) by $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$ in $d_{\mathcal{G}}$ ($d_{\mathcal{H}}$) do not occur in $d_{\mathcal{H}}$ ($d_{\mathcal{G}}$).

If $d_{\mathcal{G}}$ ends with a rule application where the principal formula is not an occurrence of φ on the opposite side to $d_{\mathcal{H}}$, then (as in the propositional case), we can make use of the almost-substitutivity of the rule and apply the induction hypothesis. The assumption that new free variables are distinct in $d_{\mathcal{G}}$ from those in $d_{\mathcal{H}}$ and vice versa ensures that this also works for the quantifier rules. Let us therefore assume – since propositional connectives are treated in the proof of Theorem 4.1.1 – that φ is of the form $(\forall x)\psi(x)$ and $d_{\mathcal{G}}$ ends with

$$\frac{\mathcal{G}' \mid \Gamma, [(\forall x)\psi(x)]^{\lambda-1}, \psi(t) \Rightarrow \Delta}{\mathcal{G}' \mid \Gamma, [(\forall x)\psi(x)]^{\lambda} \Rightarrow \Delta} \quad \text{or} \quad \frac{\mathcal{G}' \mid \Gamma \Rightarrow \psi(a), [(\forall x)\psi(x)]^{\lambda-1}, \Delta}{\mathcal{G}' \mid \Gamma \Rightarrow [(\forall x)\psi(x)]^{\lambda}, \Delta}$$

where $\varphi \notin \Gamma \uplus \Delta$ and $\mathcal{H}$ is of the form, respectively

$$\mathcal{H}' \mid \Pi \Rightarrow \underline{(\forall x)\psi(x)}, \Sigma \qquad \text{or} \qquad \mathcal{H}' \mid \Pi, \underline{(\forall x)\psi(x)} \Rightarrow \Sigma.$$

Let $\mathcal{G}^{\mathcal{H}} \in \mathrm{CUT}(\mathcal{G}, \mathcal{H})$. The only tricky case is when $\mathcal{G}^{\mathcal{H}}$ is of the form $(\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda} \Rightarrow \Sigma^{\lambda}, \Delta)$ where $\mathcal{G}'' \in \mathrm{CUT}(\mathcal{G}', \mathcal{H})$. But then also

$$\text{either} \quad \frac{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1}, \psi(t) \Rightarrow \Sigma^{\lambda-1}, \Delta}{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1}, (\forall x)\psi(x) \Rightarrow \Sigma^{\lambda-1}, \Delta}$$

$$\text{or} \quad \frac{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1} \Rightarrow \psi(a), \Sigma^{\lambda-1}, \Delta}{\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1} \Rightarrow (\forall x)\psi(x), \Sigma^{\lambda-1}, \Delta}$$

is an instance of the appropriate rule. Moreover, by the induction hypothesis, the premise is derivable so we have a derivation d ending with such a rule application.

If $\mathrm{e}(d_{\mathcal{H}}) = 1$, i.e., $d_{\mathcal{H}}$ does not end with the application of a logical rule to the marked occurrence of φ, then we proceed as in the propositional case. Suppose therefore that $\mathrm{e}(d_{\mathcal{H}}) = 0$: i.e., $d_{\mathcal{H}}$ ends with an application of $(\forall\Rightarrow)$ or $(\Rightarrow\forall)$ to the marked occurrence of φ, and is of the form:

$$\frac{\mathcal{H}' \mid \Pi \Rightarrow \psi(a), \Sigma}{\mathcal{H}' \mid \Pi \Rightarrow (\forall x)\psi(x), \Sigma} \quad \text{or} \quad \frac{\mathcal{H}' \mid \Pi, \psi(t) \Rightarrow \Sigma}{\mathcal{H}' \mid \Pi, (\forall x)\psi(x) \Rightarrow \Sigma}$$

By Lemma 6.1.2, there is a derivation of the same height of $(\mathcal{H}' \mid \Pi \Rightarrow \psi(t), \Sigma)$ or $(\mathcal{H}' \mid \mathcal{G}'' \mid \Gamma, \Pi^{\lambda-1} \Rightarrow \psi(t), \Sigma^{\lambda-1}, \Delta)$. But then by the induction hypothesis, since $\mathrm{cp}(\psi(t)) < \mathrm{cp}((\forall x)\psi(x))$, using a further application of (EC), $\vdash_{\mathrm{GL}^\circ} \mathcal{G}^{\mathcal{H}}$. □

It follows from cut elimination that GL$\forall$ is a conservative extensions of GL. Just observe that any GL$\forall$-derivable propositional hypersequent has a cut-free derivation in GL$\forall$ which does not involve any quantifiers. It is also straightforward to extend density elimination to first-order Gentzen systems: the quantifier rules are treated just like the rules for other connectives. Moreover, density elimination can be used to establish standard completeness results for first-order fuzzy logics following essentially the same procedure as for propositional fuzzy logics described in Section 4.

Cut elimination will not help us with decidability; indeed, all these logics are undecidable. However, we can use these results instead to prove versions of Herbrand's theorem and Skolemization for their prenex fragments. Recall that a *prenex formula* is a first-order formula with all the quantifiers at the front. We can show that the validity of a prenex formula is equivalent to the validity of a set of propositional formulas. The key technical tool for establishing this result is a "mid-hypersequent theorem" (an analogue of Gentzen's mid-sequent theorem) for a calculus GL$\forall$, stating that any GL$\forall$-derivable prenex formula has a GL$\forall$-derivation where the propositional connective inferences precede all quantifier inferences. It then follows that there exist hypersequents in the derivation with no propositional connective inference below or quantifier inference above.

For clarity, we focus on the case of GUL$\forall$, and to save us some effort, perform manipulations on a slight variant of this calculus. Let GUL$\forall^{\vee}$ be the cut-free hypersequent calculus GUL$\forall^{\circ}$ with (ID) restricted to instances containing only variables, and $(\vee\Rightarrow)$ and $(\wedge\Rightarrow)$ replaced with

$$\frac{\mathcal{G} \mid \Gamma_1, \varphi \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2, \psi \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \varphi \vee \psi \Rightarrow \Delta_1 \mid \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_2}\ (\vee\Rightarrow)^{\vee} \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \psi}{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi \wedge \psi \mid \Gamma_2 \Rightarrow \varphi \wedge \psi}\ (\Rightarrow\wedge)^{\vee}$$

These rules are easily derived in GUL$\forall$ using (COM), and we leave it as an (easy) exercise to show that $\vdash_{\mathrm{GUL}\forall} \mathcal{G}$ iff $\vdash_{\mathrm{GUL}\forall^{\vee}} \mathcal{G}$.

THEOREM 6.1.4. *Let $\mathcal{G}$ be a single-conclusion hypersequent containing only prenex formulas. If $d \vdash_{\mathrm{GUL}\forall^{\vee}} \mathcal{G}$, then $d' \vdash_{\mathrm{GUL}\forall^{\vee}} \mathcal{G}$ for some derivation d' where no propositional inference is below a quantifier inference.*

Proof. Let the order $o(d)$ of a derivation d be the multiset containing the lengths (cardinalities) of any sub-branches of d (i.e., subpaths of branches of d) that start with a propositional connective inference and end with a quantifier inference. Then it is sufficient to establish the following:

If $d \vdash_{\mathrm{GUL}\forall^{\vee}} \mathcal{G}$, then $d' \vdash_{\mathrm{GUL}\forall^{\vee}} \mathcal{G}$ for some derivation d' where $o(d') = []$.

We proceed by induction on $o(d)$ using the standard multiset ordering $<$. The base case where $o(d) = []$ is immediate. For the inductive step we have a number of possibilities. (1) Suppose that a quantifier inference occurs directly above a propositional connective inference. Then we can rearrange the derivation so that the quantifier rule application is below the propositional connective rule application, and use the induction hypothesis. The only tricky cases are where $(\Rightarrow\exists)$ appears above $(\vee\Rightarrow)^{\vee}$, or $(\forall\Rightarrow)$ above $(\Rightarrow\wedge)^{\vee}$. Let us consider the former, i.e., d ends with:

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{H} \mid \Gamma_1, \varphi \Rightarrow \chi(t), \Delta_1}}{\mathcal{H} \mid \Gamma_1, \varphi \Rightarrow (\exists x)\chi(x), \Delta_1}\,(\Rightarrow\exists) \quad \dfrac{\vdots\, d_2}{\mathcal{H} \mid \Gamma_2, \psi \Rightarrow \Delta_2}}{\mathcal{H} \mid \Gamma_1, \varphi \vee \psi \Rightarrow (\exists x)\chi(x), \Delta_1 \mid \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_2}\,(\vee\Rightarrow)^{\vee}$$

We can replace this with the following derivation d':

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{H} \mid \Gamma_1, \varphi \Rightarrow \chi(t), \Delta_1} \quad \dfrac{\vdots\, d_2}{\mathcal{H} \mid \Gamma_2, \psi \Rightarrow \Delta_2}}{\mathcal{H} \mid \Gamma_1, \varphi \vee \psi \Rightarrow \chi(t), \Delta_1 \mid \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_2}\,(\vee\Rightarrow)^{\vee}}{\mathcal{H} \mid \Gamma_1, \varphi \vee \psi \Rightarrow (\exists x)\chi(x), \Delta_1 \mid \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_2}\,(\Rightarrow\exists)$$

Since $o(d') < o(d)$ the induction hypothesis can be applied.

If the previous case does not occur, then there must be a sub-branch with structural inferences occurring between the propositional connective inference and quantifier inferences. We note without proof that applications of (EC) can be pushed downwards over the other structural rules in derivations, and (EW) can be pushed upwards. If a quantifier inference is directly above (EW) or (COM), then we can move the application of the structural rule above the quantifier inference. In the (most complicated) case of (COM) and ($\forall\Rightarrow$), we have the following situation:

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{H} \mid \Gamma_1, \psi(t), \Pi_1 \Rightarrow \Sigma_1, \Delta_1}}{\mathcal{H} \mid \Gamma_1, (\forall x)\psi(x), \Pi_1 \Rightarrow \Sigma_1, \Delta_1}\,(\forall\Rightarrow) \quad \dfrac{\vdots\, d_2}{\mathcal{H} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}}{\mathcal{H} \mid \Gamma_1, (\forall x)\psi(x), \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\,(\text{COM})$$

We can replace this with the following derivation d':

$$\dfrac{\dfrac{\dfrac{\vdots\, d_1}{\mathcal{H} \mid \Gamma_1, \psi(t), \Pi_1 \Rightarrow \Sigma_1, \Delta_1} \quad \dfrac{\vdots\, d_2}{\mathcal{H} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}}{\mathcal{H} \mid \Gamma_1, \psi(t), \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\,(\text{COM})}{\mathcal{H} \mid \Gamma_1, (\forall x)\psi(x), \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}\,(\forall\Rightarrow)$$

Since $o(d') < o(d)$ the induction hypothesis can be applied.

The final possibility is that an application of (EC) occurs directly above a logical connective inference. Again we are always able to push the relevant rule application upwards. □

For a formula φ, let C_φ, F_φ, and P_φ be, respectively, the constants, non-nullary function symbols, and predicate symbols occurring in φ, adding a constant to C_φ if it is empty. The *Herbrand universe* of φ is defined as $H(\varphi) = \bigcup_{n=0}^{\infty} H_n(\varphi)$ where

$$\begin{array}{rcl} H_0(\varphi) & = & C_\varphi \\ H_{n+1}(\varphi) & = & H_n(\varphi) \cup \{f(t_1, \ldots, t_k) \mid t_1, \ldots, t_k \in H_n(\varphi) \text{ and } f \in F_\varphi \text{ with arity } k\}. \end{array}$$

Suppose now that $\vdash_{\mathrm{GUL}\forall} \Rightarrow (\exists\bar{x})\varphi(\bar{x})$ where φ is quantifier-free. Then by Theorem 6.1.4, we have a $\mathrm{GUL}\forall^{\vee}$-derivation of $(\Rightarrow (\exists\bar{x})\varphi(\bar{x}))$ where no propositional inference follows a quantifier inference. So there must be a $\mathrm{GUL}\forall^{\vee}$-derivable "mid-hypersequent" $(\Rightarrow \varphi(\bar{t}_1) \mid \ldots \mid\Rightarrow \varphi(\bar{t}_n))$ for some $\bar{t}_1, \ldots, \bar{t}_n \in H(\varphi)$, and hence:

THEOREM 6.1.5. *$\vdash_{\mathrm{GUL}\forall} \Rightarrow (\exists\bar{x})\varphi(\bar{x})$ iff $\vdash_{\mathrm{GUL}\forall} \Rightarrow \bigvee_{i=1}^{n} \varphi(\bar{t}_i)$ for some terms $\bar{t}_1, \ldots, \bar{t}_n \in H(\varphi)$.*

Herbrand's theorem allows us to reduce the validity problem for an existential formula to the validity of formulas that are (essentially) propositional. We can also make use of this theorem (as in classical logic) to do the same for prenex formulas. The idea here is to remove universal quantifiers iteratively, replacing the variables that they bind with terms consisting of a new function symbol with variables bound by preceding existential quantifiers. This process is called Skolemization for fuzzy logics, although it is worth noting that for classical logic, the usual process involves removing existential quantifiers and preserving satisfiability.

Let φ be a prenex formula and assume harmlessly that the ith occurrence of $\forall$ is labelled $\forall^i$ and that no function symbol f_i of any arity occurs in φ. Then the *Skolem form* φ^S of φ is defined by induction as follows:

1. If φ is of the form $(\exists\bar{x})\psi(\bar{x})$ where ψ is quantifier-free, then φ^S is $(\exists\bar{x})\psi(\bar{x})$.
2. If φ is of the form $(\exists\bar{x})(\forall^i y)\psi(\bar{x}, y)$, then φ^S is $((\exists\bar{x})\psi(\bar{x}, f_i(\bar{x})))^S$.

EXAMPLE 6.1.6. *Consider the prenex formula $\varphi = (\exists x)(\forall y)(\exists z)(\forall u)p(x, y, z, u)$. Skolemizing in two steps, we obtain that $\varphi^S = (\exists x)(\exists z)p(x, f(x), z, g(x, z))$.*

Focussing again for clarity on the case of $\mathrm{GUL}\forall$, note that $\vdash_{\mathrm{GUL}\forall} \Rightarrow \varphi \to \varphi^S$ follows inductively since $\vdash_{\mathrm{GUL}\forall} \Rightarrow (\exists\bar{x})(\forall y)\psi(\bar{x}, y) \to (\exists\bar{x})\psi(\bar{x}, f(\bar{x}))$. The other direction does not hold in general, even in classical logic. However, we can show instead the weaker claim that $\vdash_{\mathrm{GUL}\forall} \Rightarrow \varphi^S$ implies $\vdash_{\mathrm{GUL}\forall} \Rightarrow \varphi$.

LEMMA 6.1.7. *Let $(\exists\bar{x})\psi(\bar{x})$ be the Skolem form of a prenex formula φ, and $\bar{t}_1, \ldots, \bar{t}_n \in H((\exists\bar{x})\psi(\bar{x}))$. Then $\Rightarrow \varphi$ is derivable from $\Rightarrow \psi(\bar{t}_1) \mid \ldots \mid\Rightarrow \psi(\bar{t}_n)$ using* (EW), (EC), $(\Rightarrow\forall)$, *and* $(\Rightarrow\exists)$, *where in* $(\Rightarrow\forall)$, *any variable-free term not occurring in the conclusion may be used in the premise.*

Hence if $(\exists\bar{x})\psi(\bar{x})$ is the Skolem form of a prenex formula φ and, moreover, $\vdash_{\mathrm{GUL}\forall} \Rightarrow \bigvee_{i=1}^{n} \psi(\bar{t}_i)$ for some $\bar{t}_1, \ldots, \bar{t}_n \in H(\varphi)$, then by Lemma 6.1.7, $(\Rightarrow \varphi)$ is derivable from $(\Rightarrow \psi(\bar{t}_1) \mid \ldots \mid \psi(\bar{t}_n))$ using (EW), (EC), $(\Rightarrow\forall)$, and $(\Rightarrow\exists)$. So $\vdash_{\mathrm{GUL}\forall} \Rightarrow \varphi$ as required.

THEOREM 6.1.8. *Let $(\exists\bar{x})\psi(\bar{x})$ be the Skolem form of a prenex formula φ. Then the following are equivalent:*

(1) $\vdash_{\mathrm{GUL}\forall} \Rightarrow \varphi$

(2) $\vdash_{\mathrm{GUL}\forall} \Rightarrow (\exists\bar{x})\psi(\bar{x})$

(3) $\vdash_{\mathrm{GUL}\forall} \Rightarrow \bigvee_{i=1}^{n} \psi(\bar{t}_i)$ *for some* $\bar{t}_1, \ldots, \bar{t}_n \in H(\psi)$.

The first-order situation for Łukasiewicz logic and its relatives P and CHL is more complicated. For these logics, the set of valid formulas is not recursively enumerable (for P, not even arithmetical), so finite sets of axiom and rule schema cannot be enough. Let us take a closer look at this problem for first-order Łukasiewicz logic Ł∀ in the language with connectives ∀, ∃, →, and ⊥. Our first observation is that the Herbrand theorem cannot hold for this logic. Note that $\models_{Ł} (\exists x)p(x) \to (\exists y)p(y)$. So using quantifier-shifting equivalences of Ł∀

$$\models_{Ł} (\exists y)(\forall x)(p(x) \to p(y))$$

and by the easy direction of Skolemization

$$\models_{Ł} (\exists y)(p(f(y)) \to p(y)).$$

If the Herbrand theorem did hold for Ł∀, then for some constant c and $n \in \mathsf{N}^+$:

$$\models_{Ł} \bigvee_{i=1}^{n} (p(f^i(c)) \to p(f^{i-1}(c)))$$

where $f^0(c) = c$ and $f^{i+1}(c) = f(f^i(c))$ for all $i \in \mathsf{N}$. But now we can define a structure such that the value of $p(f^i(c))$ is i/n for $i = 0 \ldots n$, a contradiction. So the Herbrand theorem must fail.

Take another look at the formula $\bigvee_{i=1}^{n}(p(f^i(c)) \to p(f^{i-1}(c)))$, however. Although this is not a valid formula of Ł∀, it comes within "one nth" of being one. Observe that for any $r_0, r_1, \ldots, r_n \in [0, 1]$:

$$\min_{i \in \{1,\ldots,n\}} \{r_{i-1} - r_i\} \leq 1/n \quad \text{and so also} \quad \max_{i \in \{1,\ldots,n\}} \{1 - r_{i-1} + r_i\} \geq 1 - 1/n.$$

Let us write for $r \in [0, 1]$:

$$\models_{Ł}^{>r} \varphi \quad \text{iff} \quad \varphi \text{ takes a value } > r \text{ in all structures.}$$

Then for any $r < 1 - 1/n$:

$$\models_{Ł}^{>r} \bigvee_{i=1}^{n} (p(f^i(c)) \to p(f^{i-1}(c))).$$

I.e., we have "Herbrand approximations" of $(\exists y)(p(f(y)) \to p(y))$ taking values arbitrarily close to 1. This illustrates a more general phenomenon, captured by the following approximate Herbrand theorem:

THEOREM 6.1.9. $\models_{Ł} (\exists \bar{x})\varphi(\bar{x})$ *iff for all* $r < 1$*:*

$$\models_{Ł}^{>r} \bigvee_{i=1}^{n} \varphi(\bar{t}_i) \quad \textit{for some } \bar{t}_1, \ldots, \bar{t}_n \in H(\varphi).$$

This approximate Herbrand theorem has a nice corollary. Let $\varphi = (\forall\bar{x})(\exists\bar{y})\psi(\bar{x},\bar{y})$ where ψ is both quantifier-free and function-free. Then $\models_{Ł} \varphi$ iff $\models_{Ł} (\exists\bar{y})\psi(\bar{c},\bar{y})$ for some new constants $\bar{c}$. Let C be the (finite) set of constants occurring in $(\exists\bar{y})\psi(\bar{c},\bar{y})$, adding one if the set is empty. Using the previous theorem:

$$\begin{array}{lll} \models_{Ł} \varphi & \text{iff} & \text{for all } r < 1\text{:}\ \ \models_{Ł}^{>r} \bigvee_{i=1}^{n} \psi(\bar{c},\bar{t}_i)\ \text{ for some } \bar{t}_1,\ldots,\bar{t}_n \in C \\ & \text{iff} & \models_{Ł} \bigvee_{\bar{d}\in C} \psi(\bar{c},\bar{d}). \end{array}$$

But the validity problem for propositional Łukasiewicz logic is decidable, so the validity problem for function-free formulas $(\forall\bar{x})(\exists\bar{y})\psi(\bar{x},\bar{y})$ and also the validity problem for function-free one-bound-variable formulas of Ł$\forall$ are decidable.

Now we can use the approximate Herbrand theorem to establish Skolemization for Ł$\forall$. Since any formula has an equivalent prenex formula, both Skolemization and the approximate Herbrand theorem hold for all formulas of Ł$\forall$.

THEOREM 6.1.10. *Let φ be a formula and $(\mathsf{Q}\bar{y})\psi(\bar{y})$ an equivalent prenex form for φ. Let $(\exists\bar{x})\psi^F(\bar{x})$ be the Skolem form of $(\mathsf{Q}\bar{y})\psi(\bar{y})$. Then the following are equivalent:*

(1) $\models_{Ł} \varphi$

(2) $\models_{Ł} (\mathsf{Q}\bar{y})\psi(\bar{y})$

(3) $\models_{Ł} (\exists\bar{x})\psi^F(\bar{x})$

(4) For all $r < 1$: $\models_{Ł}^{>r} \bigvee_{i=1}^{n} \psi^F(\bar{t}_i)$ *for some* $\bar{t}_1,\ldots,\bar{t}_n \in H(\psi^F)$.

Now consider the system GŁ$\forall$, obtained by extending GŁ with the standard rules for the universal and existential quantifiers. It is easily seen that the standard quantifier axioms are derivable; e.g.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\psi(a)\Rightarrow\psi(a)}\,(\textsc{id}) \qquad \dfrac{}{\varphi\Rightarrow\varphi}\,(\textsc{id})}{\psi(a),\varphi\Rightarrow\varphi,\psi(a)}\,(\textsc{mix})}{\varphi\rightarrow\psi(a),\varphi\Rightarrow\psi(a)}\,(\rightarrow\Rightarrow)_{\mathrm{A}}}{(\forall x)(\varphi\rightarrow\psi),\varphi\Rightarrow\psi(a)}\,(\forall\Rightarrow)}{(\forall x)(\varphi\rightarrow\psi),\varphi\Rightarrow(\forall x)\psi}\,(\Rightarrow\forall)}{(\forall x)(\varphi\rightarrow\psi)\Rightarrow\varphi\rightarrow(\forall x)\psi}\,(\Rightarrow\rightarrow)}{\Rightarrow(\forall x)(\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow(\forall x)\psi)}\,(\Rightarrow\rightarrow)$$

GŁ$\forall$ is sound with respect to the standard semantics for Ł$\forall$; i.e., if $d \vdash_{\mathrm{GŁ}\forall} \Rightarrow \varphi$, then $\models_{Ł} \varphi$. The other direction cannot hold. On the other hand, GŁ$\forall$ extended with the cut rule (CUT) is complete with respect to the algebraic semantics defined in terms of MV-algebras. Unfortunately, cut elimination fails for GŁ$\forall$ + (CUT); consider, e.g.

$$
\dfrac{\dfrac{}{(\forall z)p(z) \Rightarrow (\forall z)p(z)}\,(\text{ID}) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a) \Rightarrow p(a)}\,(\text{ID})}{(\forall z)p(z) \Rightarrow p(a)}\,(\forall\Rightarrow) \quad \dfrac{\dfrac{}{p(a) \Rightarrow p(a)}\,(\text{ID}) \quad \dfrac{\dfrac{}{p(b) \Rightarrow p(b)}\,(\text{ID})}{(\forall z)p(z) \Rightarrow p(b)}\,(\forall\Rightarrow)}{(\forall z)p(z), p(a) \Rightarrow p(b), p(a)}\,(\text{MIX})}{(\forall z)p(z) \Rightarrow p(a) \to p(b), p(a)}\,(\Rightarrow\to)_{\text{Ł}}}{(\forall z)p(z) \Rightarrow (\forall y)(p(a) \to p(y)), p(a)}\,(\Rightarrow\forall)}{(\forall z)p(z) \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), p(a)}\,(\Rightarrow\exists)}{(\forall z)p(z) \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), (\forall z)p(z)}\,(\Rightarrow\forall)}{(\forall z)p(z) \to (\forall z)p(z) \Rightarrow (\exists x)(\forall y)(p(x) \to p(y))}\,(\to\Rightarrow)_{\text{A}}}{\Rightarrow (\exists x)(\forall y)(p(x) \to p(y))}\,(\text{CUT})
$$

where the left premise of (CUT) is

$$
\dfrac{\dfrac{}{(\forall z)p(z) \Rightarrow (\forall z)p(z)}\,(\text{ID})}{\Rightarrow (\forall z)p(z) \to (\forall z)p(z)}\,(\Rightarrow\to)
$$

No GŁ∀-derivation exists for this sequent. A simple induction shows that in such a derivation there would be a branch where any hypersequent $\mathcal{G}$ satisfies the following property: $\mathcal{G}$ contains a sequent with an occurrence of a subformula of $(\exists x)(\forall y)(p(x) \to p(y))$ on the right that does not occur on the left in any sequent in $\mathcal{G}$. But then the branch could not contain an initial sequent, a contradiction.

To obtain a calculus that is complete for the full first-order Łukasiewicz logic, we need an *infinitary* rule. Suppose that $\models_{\text{Ł}} (\exists \bar{x})\varphi(\bar{x})$. Then for all $k \in \mathsf{N}^+$: $\vdash_{\text{GŁ}\forall} \bot \Rightarrow [\psi]^k$ where $\psi = \bigvee_{i=1}^{n} \varphi^F(\bar{t}_i)$ for some $\bar{t}_1, \ldots, \bar{t}_n \in H(\varphi^F)$. So using Lemma 6.1.7 k times, we can apply the quantifier and structural rules of GŁ∀ to derive $(\bot \Rightarrow [\varphi]^k)$ from $(\bot \Rightarrow [\psi]^k)$.

THEOREM 6.1.11. *Let φ be a prenex formula. Then $\models_{\text{Ł}} \varphi$ iff φ is derivable in* GŁ∀ *extended with the rule:*

$$
\dfrac{\bot \Rightarrow [\varphi]^k \ \text{ for all } k \in \mathsf{N}^+}{\Rightarrow \varphi}
$$

EXAMPLE 6.1.12. *Consider our earlier problematic formula $(\exists x)(\forall y)(p(x) \to p(y))$. To derive this in our calculus we would have to perform an infinite number of derivations in* GŁ∀. *E.g., in the case where $n = 2$, we have:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow p(b) \to p(c)}\,(\bot\Rightarrow)_{\text{Ł}} \quad \dfrac{\dfrac{\dfrac{}{\bot \Rightarrow p(b)}\,(\bot\Rightarrow)_{\text{Ł}} \quad \dfrac{\dfrac{}{\bot \Rightarrow p(c)}\,(\bot\Rightarrow)_{\text{Ł}} \quad \dfrac{}{p(b) \Rightarrow p(b)}\,(\text{ID})}{\bot, p(b) \Rightarrow p(b), p(c)}\,(\text{MIX})}{\bot \Rightarrow p(b), p(b) \to p(c)}\,(\Rightarrow\to)_{\text{Ł}}}{\bot, p(a) \Rightarrow p(b), p(b) \to p(c)}\,(\text{W})}{\bot \Rightarrow p(a) \to p(b), p(b) \to p(c)}\,(\Rightarrow\to)_{\text{Ł}}}{\bot \Rightarrow p(a) \to p(b), (\forall y)(p(b) \to p(y))}\,(\Rightarrow\forall)}{\bot \Rightarrow p(a) \to p(b), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\exists)
$$

$$
\dfrac{\dfrac{\bot \Rightarrow p(a) \to p(b), (\exists x)(\forall y)(p(x) \to p(y))}{\bot \Rightarrow (\forall y)(p(a) \to p(y)), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\forall)}{\bot \Rightarrow (\exists x)(\forall y)(p(x) \to p(y)), (\exists x)(\forall y)(p(x) \to p(y))}\,(\Rightarrow\exists)
$$

6.2 Propositional quantifiers

The addition of propositional quantifiers $\forall p$ and $\exists p$ – intuitively "for all propositions p" and "for some proposition p" – can be useful for expressing properties of truth value sets. For example, rather than expressing density using a rule, we might use "density axioms"

$$(\forall p)((\varphi \to p) \vee (p \to \psi)) \to (\varphi \to \psi)$$

where p does not occur in φ or ψ. More generally, this extra flexibility can be used to express topological properties of sets of real numbers (the intended semantics of a logic) such as the existence of limit points or successor elements.

In classical logic, propositional quantifiers are little more than a notational convenience: $(\forall p)\varphi(p)$ and $(\exists p)\varphi(p)$ are interpreted by infima and suprema over the truth value set $\{0, 1\}$ and can therefore be defined as just $\varphi(\bot) \wedge \varphi(\top)$ and $\varphi(\bot) \vee \varphi(\top)$. However, in non-classical logics propositional quantifiers can increase the expressive power of the language quite considerably, an important example being quantified intuitionistic logic. For fuzzy logics, we will consider only proof-theoretic methods for the already tricky case of quantified Gödel logic. Let us make use of the language $\mathcal{L}_c = \{\wedge, \vee, \to, \bot, \top\}$ and add to the definition of a formula the condition that if φ is a formula and p is a variable, then $(\forall p)\varphi$ and $(\exists p)\varphi$ are formulas. As in the first-order case, we distinguish between bound variables, denoted p, q, and free variables, denoted a, b. We also write $\varphi(\bar{a})$ to denote that the free variables of φ are among those in $\bar{a}$, using $\varphi(\psi)$ to denote distinguished occurrences of a formula ψ in φ.

G-valuations are extended to such formulas as follows, where for a valuation v and $\alpha \in [0, 1]$, $v[\alpha/p]$ is defined by $v[\alpha/p](q) = \alpha$ if $q = p$ and $v[\alpha/p](q) = v(q)$ otherwise:

$$\begin{aligned} v((\exists p)\varphi) &= \sup\{v[\alpha/p](\varphi) \mid \alpha \in [0, 1]\} \\ v((\forall p)\varphi) &= \inf\{v[\alpha/p](\varphi) \mid \alpha \in [0, 1]\}. \end{aligned}$$

A formula φ is QG-*valid*, written $\models_{\mathrm{QG}} \varphi$, if $v(\varphi) = 1$ for all G-valuations v.

A Hilbert system HQG for QG is obtained as an extension of an axiomatization for Gödel logic with the axioms and rules:

(Q∃)	$\varphi(\psi) \to (\exists q)\varphi(q)$	
(∀Q)	$(\forall q)\varphi(q) \to \varphi(\psi)$	
(Q∨)	$(\forall q)(\varphi \vee \psi) \to (\varphi \vee (\forall q)\psi)$	q not occurring in φ
(QD)	$(\forall q)((\varphi \to q) \vee (q \to \psi)) \to (\varphi \to \psi)$	q not occurring in φ or ψ

$$\frac{\varphi(a) \to \psi}{(\exists q)\varphi(q) \to \psi}\ (\exists\mathrm{Q}) \qquad \frac{\varphi \to \psi(a)}{\varphi \to (\forall q)\psi(q)}\ (\mathrm{Q}\forall)$$

where a does not occur in φ or ψ.

It is straightforward to check that HQG is sound with respect to QG-validity. However, the crucial result for HQG is the following (we refer to the historical remarks for references to the rather involved proof):

THEOREM 6.2.1. HQG *admits* quantifier elimination*: for every formula φ there is a quantifier-free formula ψ whose variables occur in φ such that* $\vdash_{\mathrm{HQG}} \varphi \leftrightarrow \psi$.

Using the completeness of the ordinary axiomatization for Gödel logic, $\models_{\mathrm{QG}} \psi$ iff $\vdash_{\mathrm{HQG}} \psi$ for any quantifier-free formula ψ. Hence, combining this with quantifier elimination, we obtain $\vdash_{\mathrm{HQG}} \varphi$ iff $\models_{\mathrm{QG}} \varphi$. In fact it follows from the proof of Theorem 6.2.1

that this last result holds for HQG even when the ψ in $(\forall\mathrm{Q})$ and $(\mathrm{Q}\exists)$ are restricted to quantifier-free formulas. Moreover, quantifier elimination provides an easy proof of the interpolation property for Gödel logic.

COROLLARY 6.2.2. *Gödel logic* G *admits* interpolation*, i.e., if* $\models_{\mathrm{G}} \varphi \to \psi$*, then* $\models_{\mathrm{G}} \varphi \to \chi$ *and* $\models_{\mathrm{G}} \chi \to \psi$ *for some formula* χ *whose variables occur in both* φ *and* ψ*.*

Proof. Suppose that $\models_{\mathrm{G}} \varphi(\bar{a}, \bar{b}) \to \psi(\bar{b}, \bar{c})$ where $\bar{a}$ and $\bar{b}$ and $\bar{b}$ and $\bar{c}$ are the distinct variables occurring in φ and ψ, respectively. Then easily $\models_{\mathrm{QG}} \varphi(\bar{a}, \bar{b}) \to (\exists\bar{p})\varphi(\bar{p}, \bar{b})$ and $\models_{\mathrm{QG}} (\exists\bar{p})\varphi(\bar{p}, \bar{b}) \to \psi(\bar{b}, \bar{c})$. But then by quantifier elimination, there exists a propositional formula $\chi(\bar{b})$ such that $\models_{\mathrm{QG}} (\exists\bar{p})\varphi(\bar{p}, \bar{b}) \leftrightarrow \chi(\bar{b})$ as required. □

A hypersequent calculus for quantified Gödel logic is obtained by adding rules for propositional quantifiers to the calculus GG (where substitutions for variables are restricted to quantifier-free formulas) and, crucially in this case, the density rule. That is, GQG consists of GG extended with:

$$\frac{\mathcal{G} \mid \Gamma, \varphi(\psi) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall p)\varphi(p) \Rightarrow \Delta}\ (\forall\Rightarrow)_{\mathrm{Q}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a)}{\mathcal{G} \mid \Gamma \Rightarrow (\forall p)\varphi(p)}\ (\Rightarrow\forall)_{\mathrm{Q}}$$

$$\frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists p)\varphi(p) \Rightarrow \Delta}\ (\exists\Rightarrow)_{\mathrm{Q}} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(\psi)}{\mathcal{G} \mid \Gamma \Rightarrow (\exists p)\varphi(p)}\ (\Rightarrow\exists)_{\mathrm{Q}}$$

$$\frac{\mathcal{G} \mid \Gamma_1, a \Rightarrow \Delta \mid \Gamma_2 \Rightarrow a}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\textsc{density})$$

where ψ is quantifier-free in $(\forall\Rightarrow)_{\mathrm{Q}}$ and $(\Rightarrow\exists)_{\mathrm{Q}}$, and a does not occur in the premises of $(\Rightarrow\forall)_{\mathrm{Q}}$ or $(\exists\Rightarrow)_{\mathrm{Q}}$ or the conclusion of (DENSITY).

EXAMPLE 6.2.3. *We illustrate this calculus with a derivation of the density axioms (for space reasons, omitting the easy derivations of* $\varphi, \varphi \to a \Rightarrow a$ *and* $a, a \to \psi \Rightarrow \psi$*):*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \to a, \varphi \Rightarrow a \qquad \dfrac{\dfrac{\varphi, \varphi \to a \Rightarrow a \quad a \to \psi, a \Rightarrow \psi}{a \to \psi, \varphi \Rightarrow a \mid \varphi \to a, a \Rightarrow \psi}\ (\textsc{com}) \qquad a \to \psi, a \Rightarrow \psi}{a \to \psi, \varphi \Rightarrow a \mid (\varphi \to a) \vee (a \to \psi), a \Rightarrow \psi}\ (\vee\Rightarrow)^*}{(\varphi \to a) \vee (a \to \psi), \varphi \Rightarrow a \mid (\varphi \to a) \vee (a \to \psi), a \Rightarrow \psi}\ (\vee\Rightarrow)^*}{(\varphi \to a) \vee (a \to \psi), \varphi \Rightarrow a \mid (\forall q)((\varphi \to q) \vee (q \to \psi)), a \Rightarrow \psi}\ (\forall\Rightarrow)_Q}{(\forall q)((\varphi \to q) \vee (q \to \psi)), \varphi \Rightarrow a \mid (\forall q)((\varphi \to q) \vee (q \to \psi)), a \Rightarrow \psi}\ (\forall\Rightarrow)_Q}{(\forall q)((\varphi \to q) \vee (q \to \psi)), (\forall q)((\varphi \to q) \vee (q \to \psi)), \varphi \Rightarrow \psi}\ (\textsc{density})}{(\forall q)((\varphi \to q) \vee (q \to \psi)), \varphi \Rightarrow \psi}\ (\textsc{c})}{(\forall q)((\varphi \to q) \vee (q \to \psi)) \Rightarrow \varphi \to \psi}\ (\Rightarrow\to)}{\Rightarrow (\forall q)((\varphi \to q) \vee (q \to \psi)) \to (\varphi \to \psi)}\ (\Rightarrow\to)$$

Notice that the contraction rule (C) *is essential for this derivation. To find axiomatizations of quantified fuzzy logics lacking contraction, alternative density axioms are required, or the presence of the density rule itself.*

Soundness and completeness proofs for GQG follow the usual pattern. It is easy to see that the quantifier rules preserve validity, so an induction on the height of a derivation gives soundness, while completeness follows as before from the fact that the extra axioms of HQG (restricting the ψ in (∀Q) and (Q∃) to quantifier-free formulas) are derivable, and the extra rules are admissible.

THEOREM 6.2.4. $\vdash_{\mathrm{GQG}} \mathcal{G}$ *iff* $\vdash_{\mathrm{QG}} \mathrm{I}(\mathcal{G})$.

Cut elimination also follows the same pattern as described above but here there are a couple of important extra points to consider.

THEOREM 6.2.5. *Cut elimination holds for* GQG.

Proof. As in Theorem 4.1.1, it suffices to prove that for any hypersequent $\mathcal{G}$ and hypersequent $\mathcal{H}$ with marked formula φ:

$$\text{If } d_{\mathcal{G}} \vdash_{\mathrm{GQG}^\circ} \mathcal{G} \text{ and } d_{\mathcal{H}} \vdash_{\mathrm{GQG}^\circ} \mathcal{H}, \text{ then} \vdash_{\mathrm{GQG}^\circ} \mathrm{CUT}(\mathcal{G}, \mathcal{H}).$$

However, this time we have to add an extra parameter to our induction hypothesis to cope with the fact that the complexity of the cut formula treated can increase when stepping, e.g., from $(\forall p)\psi(p)$ to $\psi(\chi)$. We let $\mathrm{q}(\varphi)$ be the number of occurrences of quantifiers in φ, and prove the claim by induction on the lexicographically ordered quadruple:

$$\langle \mathrm{q}(\varphi), \mathrm{cp}(\varphi), \mathrm{e}(d_{\mathcal{H}}), \mathrm{h}(d_{\mathcal{G}})\rangle.$$

Notice first that we can assume, similarly to the first-order case, that new variables introduced by the density rule are completely new, i.e., do not occur elsewhere in the derivations of $\mathcal{G}$ and $\mathcal{H}$. Given this assumption, cases involving the density rule proceed in the same way as for other structural rules. Where we really need the extra parameter in the induction hypothesis is the case where both branches end with a quantifier rule applied to an occurrence of φ. E.g., for $\varphi = (\forall p)\psi(p)$, we might have:

$$\dfrac{\dfrac{\vdots}{\mathcal{G}' \mid \Gamma, \psi(\chi), [(\forall p)\psi(p)]^{n-1} \Rightarrow \Delta}}{\mathcal{G}' \mid \Gamma, [(\forall p)\psi(p)]^{n} \Rightarrow \Delta}\ (\forall\Rightarrow)_{\mathrm{Q}} \qquad \dfrac{\dfrac{\vdots}{\mathcal{H}' \mid \Pi \Rightarrow \psi(a)}}{\mathcal{H}' \mid \Pi \Rightarrow (\forall p)\psi(p)}\ (\Rightarrow\forall)_{\mathrm{Q}}$$

Consider a member of $\mathrm{CUT}(\mathcal{G}, \mathcal{H})$ of the form (the only tricky case):

$$\mathcal{G}'' \mid \Gamma, \Pi^n \Rightarrow \Delta$$

where $\mathcal{G}'' \in \mathrm{CUT}(\mathcal{G}', \mathcal{H})$. Then by the induction hypothesis:

$$\vdash_{\mathrm{GQG}^\circ} \mathcal{G}'' \mid \Gamma, \psi(\chi), \Pi^{n-1} \Rightarrow \Delta$$

We can substitute χ for a in the derivation of $\mathcal{H}' \mid \Pi \Rightarrow \psi(a)$ (an easy induction) to obtain a derivation of $\mathcal{H}' \mid \Pi \Rightarrow \psi(\chi)$. But $\mathrm{q}(\psi(\chi)) < \mathrm{q}((\forall p)\psi(p))$. So by the induction hypothesis again and (EC), $\vdash_{\mathrm{GQG}^\circ} \mathcal{G}'' \mid \Gamma, \Pi^n \Rightarrow \Delta$ as required. □

Of course, density elimination, which holds for GG and GG∀, does not hold for GQG since the density rule is needed to prove the (QD) axioms.

6.3 Modalities

Modal logics and their relatives such as description logics occupy an important realm between propositional and first-order classical logic. Ideally, they are expressive enough to capture and reason about notions such as time, space, knowledge, etc., but also have good computational properties such as decidability and reasonable complexity bounds. Combining modal notions with fuzziness in the form of modal fuzzy logics has been explored in a number of contexts such as representing and reasoning about fuzzy beliefs, spatial reasoning in the presence of vagueness, fuzzy similarity measures, and fuzzy description logics. General approaches have also been proposed for dealing with many-valued modal logics, although most of the axiomatization and decidability results so far have been limited to the finite-valued case. References are provided in the historical remarks at the end of this chapter.

Proof theoretic methods for modal fuzzy logics have not as yet been developed with any great degree of generality or uniformity, an exception to this being the case where a modal operator $\Box$ is interpreted as a truth stresser; e.g., $\Box\varphi$ is interpreted as φ is "definitely true", "very true", "more true than false", etc. For such modalities, where the intended interpretation of $\Box$ is a unary function on $[0,1]$ (or some other linearly ordered set), we can extend the language $\mathcal{L}_p$ with the single unary operator $\Box$ and add to our hypersequent calculi rules such as:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\varphi}\ (\Box) \qquad \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta}\ (\Box\Rightarrow) \qquad \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\varphi}\ (\Rightarrow\Box)$$

where $\Box[\varphi_1, \ldots, \varphi_n]$ stands for $[\Box\varphi_1, \ldots, \Box\varphi_n]$.

These rules are just hypersequent versions of rules familiar from sequent calculi for the modal logics K, KT, K4, and S4. The standard necessitation rule for modal logics corresponds to instances of $(\Box)$ or $(\Rightarrow\Box)$ where Γ and $\mathcal{G}$ are empty, and the usual K axioms are derivable using the core implication rules and $(\Box)$ as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\varphi \Rightarrow \varphi}\ (\text{ID}) \quad \cfrac{}{\psi \Rightarrow \psi}\ (\text{ID})}{\varphi \to \psi, \varphi \Rightarrow \psi}\ (\to\Rightarrow)}{\Box(\varphi \to \psi), \Box\varphi \Rightarrow \Box\psi}\ (\Box)}{\Box(\varphi \to \psi) \Rightarrow \Box\varphi \to \Box\psi}\ (\Rightarrow\to)}{\Rightarrow \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}\ (\Rightarrow\to)$$

It is also easy to see that T axioms of the form $\Box\varphi \to \varphi$ and 4 axioms of the form $\Box\varphi \to \Box\Box\varphi$ are derivable using $(\Box\Rightarrow)$ and $(\Rightarrow\Box)$, respectively. However, all is not quite what it seems here. "Shifting modality" axioms of the form $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$, which are not derivable even in the modal logic S4, are derivable here using $(\Box)$ together with the usual logical rules, (COM), (EW), and (EC) (the top hypersequent being derived as in Example 3.2.2):

$$\cfrac{\cfrac{\cfrac{\varphi \vee \psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}{\Box(\varphi \vee \psi) \Rightarrow \Box\varphi \mid \Box(\varphi \vee \psi) \Rightarrow \Box\psi}\ (\Box)}{\Box(\varphi \vee \psi) \Rightarrow \Box\varphi \vee \Box\psi}\ (\Rightarrow\vee)}{\Rightarrow \Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)}\ (\Rightarrow\to)$$

In fact, the derivability of these axioms is needed for the calculi to be complete with respect to linearly ordered algebras.

The behaviour of $\Box$ can be characterized further by adding "modal versions" of structural rules such as:

$$\frac{\mathcal{G} \mid \Gamma, \Box\Pi, \Box\Pi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\Pi \Rightarrow \Delta}\ (\textsc{c})_\Box \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\Pi \Rightarrow \Delta}\ (\textsc{w})_\Box \quad \frac{\mathcal{G} \mid \Box\Gamma_1, \Pi \Rightarrow \Sigma}{\mathcal{G} \mid \Box\Gamma_1, \Gamma_2 \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}\ (\textsc{split})_\Box$$

The structural rules $(\textsc{c})_\Box$ and $(\textsc{w})_\Box$ are hypersequent versions of rules used for the exponential ! in linear logic, and may be used to provide embeddings of fuzzy logics into other fuzzy logics extended with a suitable modality. The rule $(\textsc{split})_\Box$, on the other hand, is used to provide proof systems for fuzzy logics extended with the globalization modality Δ defined as $\Delta x = 1$ if $x = 1$, $\Delta x = 0$ otherwise.

Given a hypersequent calculus GL, we can define, for example, the systems

$$\begin{array}{rclrcl}
\mathrm{GLK^r} & = & \mathrm{GL} + (\Box) & \mathrm{GLKT^r} & = & \mathrm{GLK^r} + (\Box\Rightarrow) \\
\mathrm{GLK4^r} & = & \mathrm{GLK^r} + (\Rightarrow\Box) & \mathrm{GLS4^r} & = & \mathrm{GLK^r} + (\Box\Rightarrow) + (\Rightarrow\Box) \\
\mathrm{GL}_\Delta & = & \mathrm{GLS4^r} + (\textsc{split})_\Box & \mathrm{GL!} & = & \mathrm{GLS4^r} + (\textsc{c})_\Box + (\textsc{w})_\Box
\end{array}$$

where ^{r} denotes that these systems are complete with respect to linearly ordered algebras. Moreover, if GL is an extension of GUL with substitutive single-conclusion rules or extension of GIUL with substitutive rules, then the extended calculi also admit cut elimination.

In general, however, modal fuzzy logics are based not solely on functions on $[0, 1]$, but rather on some variant of the Kripke frames and models appearing in classical modal logics. Here, the challenge is to prove that a proposed Hilbert or Gentzen system is complete with respect to the given semantics. Since the only proof-theoretic results in this direction have been based on Gödel logic, we confine our attention to this case, making use of a language $\mathcal{L}_{\Box\Diamond}$ that extends $\mathcal{L}_c$ with the unary operators $\Box$ and $\Diamond$. Following general approaches described in the literature, Gödel modal logics are generalizations of the modal logic K where connectives behave locally at individual worlds as in Gödel logic. In particular, GK and $\mathrm{GK^F}$ are defined as modal Gödel fuzzy logics based on standard Kripke frames and Kripke frames with fuzzy accessibility relations, respectively.

We define a *fuzzy Kripke frame* to be a pair $F = \langle W, R\rangle$ where W is a non-empty set of *worlds* and $R\colon W \times W \to [0, 1]$ is a binary *fuzzy accessibility relation* on W. If $Rxy \in \{0, 1\}$ for all $x, y \in W$, then R is called *crisp* and F is called simply a *(standard) Kripke frame*. In this case, we may consider $R \subseteq W^2$ and write Rxy or $(x, y) \in R$ to mean $Rxy = 1$. Next, a *Kripke model for* $\mathrm{GK^F}$ is a 3-tuple $K = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is a fuzzy Kripke frame and $V\colon Fm_{\mathcal{L}_{\Box\Diamond}} \times W \to [0, 1]$ is a mapping, called a $\mathrm{GK^F}$-*valuation*, satisfying

$$\begin{array}{rcl}
V(\top, x) & = & 1 \\
V(\bot, x) & = & 0 \\
V(\varphi \to \psi, x) & = & V(\varphi, x) \to_{\mathrm{G}} V(\psi, x) \\
V(\varphi \wedge \psi, x) & = & \min(V(\varphi, x), V(\psi, x)) \\
V(\varphi \vee \psi, x) & = & \max(V(\varphi, x), V(\psi, x)) \\
V(\Box\varphi, x) & = & \inf\{Rxy \to_{\mathrm{G}} V(\varphi, y) \mid y \in W\} \\
V(\Diamond\varphi, x) & = & \sup\{\min(V(\varphi, y), Rxy) \mid y \in W\}.
\end{array}$$

A *Kripke model for* GK satisfies the extra condition that $\langle W, R\rangle$ is a standard Kripke frame. In this case, the conditions for $\Box$ and $\Diamond$ may also be read as:

$$\begin{array}{rcl} V(\Box\varphi, x) & = & \inf(\{1\} \cup \{V(\varphi, y) \mid Rxy\}) \\ V(\Diamond\varphi, x) & = & \sup(\{0\} \cup \{V(\varphi, y) \mid Rxy\}). \end{array}$$

A formula φ is *valid* in $\langle W, R, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$, and L-*valid* for $\mathrm{L} \in \{\mathrm{GK}, \mathrm{GK}^{\mathrm{F}}\}$, written $\models_{\mathrm{L}} \varphi$, if φ is valid in all Kripke models $\langle W, R, V\rangle$ for L.

Developing proof-theoretic methods for the full languages of these logics remains an open problem, as is finding axiomatizations and establishing decidability. Instead, let us consider the "box" and "diamond" fragments of GK and GK^{F} based on restrictions of the language $\mathcal{L}_{\Box\Diamond}$ to the single modality sublanguages $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\Diamond}$, respectively. Proof-theoretic methods have been used to establish decidability and complexity results for these fragments, as well as completeness for the following axiomatizations:

- $\mathrm{HGK}_{\Box}$ is HG extended with

$$\begin{array}{ll} (K_{\Box}) & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ (Z_{\Box}) & \neg\neg\Box\varphi \to \Box\neg\neg\varphi \end{array}$$

$$\frac{\varphi}{\Box\varphi}\ (\textsc{nec})_{\Box}$$

- $\mathrm{HGK}_{\Diamond}$ is HG extended with

$$\begin{array}{ll} (K_{\Diamond}) & \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi) \\ (Z_{\Diamond}) & \Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi \\ (F_{\Diamond}) & \neg\Diamond\bot \end{array}$$

$$\frac{(\varphi \to \psi) \lor \chi}{(\Diamond\varphi \to \Diamond\psi) \lor \Diamond\chi}\ (\textsc{nec})_{\Diamond}$$

- $\mathrm{HGK}^{\mathrm{F}}_{\Diamond}$ is $\mathrm{HGK}_{\Diamond}$ with $(\textsc{nec})_{\Diamond}$ replaced by

$$\frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ (\textsc{nec})^{*}_{\Diamond}$$

The diamond fragment of GK^{F} has the finite model property, and so is decidable, but this is not the case either for the diamond fragment of GK or the box fragment of GK, which coincides with the box fragment of GK^{F}.

To provide a uniform proof-theoretic presentation of these fragments, we turn to the sequent of relations framework introduced in the previous section. Let us define SG to consist of the logical rules of Figure 7 extended with the following rules:

$$\frac{}{\mathcal{G} \mid \varphi \leq \varphi}\ (\textsc{id}) \qquad \frac{\mathcal{G} \mid \top \leq \bot \mid \bot < \bot}{\mathcal{G}}\ (\textsc{cs}) \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \varphi \lhd \psi}\ (\textsc{ew})$$

$$\frac{\mathcal{G} \mid \varphi \leq \psi \mid \top \leq \psi}{\mathcal{G} \mid \varphi \leq \psi}\ (\textsc{wl}) \qquad \frac{\mathcal{G} \mid \varphi \leq \psi \mid \varphi \leq \bot}{\mathcal{G} \mid \varphi \leq \psi}\ (\textsc{wr})$$

$$\frac{\mathcal{G} \mid \varphi_1 \lhd_1 \psi_1 \mid \varphi_2 \lhd_2 \psi_2 \mid \varphi_1 \leq \psi_2 \quad \mathcal{G} \mid \varphi_1 \lhd_1 \psi_1 \mid \varphi_2 \lhd_2 \psi_2 \mid \varphi_2 \lhd_1 \psi_1}{\mathcal{G} \mid \varphi_1 \lhd_1 \psi_1 \mid \varphi_2 \lhd_2 \psi_2}\ (\textsc{com})$$

We adopt the following helpful notation for sets of relations:

$$\begin{array}{rcl}
[\varphi_1, \ldots, \varphi_n] \lhd \psi & =_{\text{def}} & \varphi_1 \lhd \psi \mid \ldots \mid \varphi_n \lhd \psi \\
[\,] \leq \psi & =_{\text{def}} & \top \leq \psi \\
[\,] < \psi & =_{\text{def}} & \emptyset \\
\varphi \lhd [\psi_1, \ldots, \psi_m] & =_{\text{def}} & \varphi \lhd \psi_1 \mid \ldots \mid \varphi \lhd \psi_m \\
\varphi \leq [\,] & =_{\text{def}} & \varphi \lhd \bot \\
\varphi < [\,] & =_{\text{def}} & \emptyset.
\end{array}$$

Note that we always restrict expressions $\Gamma \lhd \Delta$ to cases where either Γ or Δ has at most one element. We remark, moreover, that Γ and Δ can be considered as sets of formulas rather than multisets without changing the meaning of the notation.

Sequent of relations calculi for fragments of Gödel modal logics may now be defined as follows:

- $\mathrm{SGK}_\Box$ consists of SG (for $\mathcal{L}_\Box$) extended with:

$$\dfrac{\Gamma \leq \psi \mid \Pi \leq \bot}{\mathcal{G} \mid \Box\Gamma \leq \Box\psi \mid \Box\Pi \leq \bot}\ (\Box)$$

- $\mathrm{SGK}_\Diamond$ consists of SG (for $\mathcal{L}_\Diamond$) extended with:

$$\dfrac{\varphi \leq \Delta \mid \bot < \Sigma \mid \top \leq \Pi}{\mathcal{G} \mid \Diamond\varphi \leq \Diamond\Delta \mid \bot < \Diamond\Sigma \mid \top \leq \Diamond\Pi}\ (\Diamond)$$

- $\mathrm{SGK}^{\mathrm{F}}_\Diamond$ consists of SG (for $\mathcal{L}_\Diamond$) extended with:

$$\dfrac{\varphi \leq \Delta \mid \bot < \Sigma}{\mathcal{G} \mid \Diamond\varphi \leq \Diamond\Delta \mid \bot < \Diamond\Sigma}\ (\Diamond)^*$$

EXAMPLE 6.3.1. *Consider the following derivation of a sequent of relations corresponding to the axiom* $\Box(p \to q) \to (\Box p \to \Box q)$*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \leq q \mid q < p \mid p \leq p}\,(\text{ID}) \quad \dfrac{}{p \leq q \mid q < p \mid q \leq q}\,(\text{ID})}{p \leq q \mid q < p}\,(\text{COM})}{p \leq q \mid \top \leq q \mid q < p}\,(\text{EW}) \quad \dfrac{}{p \leq q \mid q \leq q}\,(\text{ID})}{p \leq q \mid p \to q \leq q}\,(\to\leq)}{\Box p \leq \Box q \mid \Box(p \to q) \leq \Box q}\,(\Box)}{\Box(p \to q) \leq \Box p \to \Box q}\,(\leq\to)}{\Box(p \to q) \leq \Box p \to \Box q \mid \top \leq \Box p \to \Box q}\,(\text{EW})}{\top \leq \Box(p \to q) \to (\Box p \to \Box q)}\,(\leq\to)$$

Soundness is relatively straightforward to establish for these calculi but completeness is rather involved. We note merely that decidability is an immediate consequence of completeness since for any given sequent of relations there is a finite number of sequent of relations that can occur in a derivation of it in one of these calculi. Moreover, with a little more analysis, the calculi can be used to show that each of the fragments is in PSPACE and hence PSPACE-complete.

We note finally that in the case of the box fragment, a hypersequent calculus $\mathrm{GGK}_{\Box}$ admitting cut elimination is obtained by extending GG with the rule

$$\frac{\Pi \Rightarrow \mid \Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Pi \Rightarrow \mid \Box\Gamma \Rightarrow \Box\varphi}\ (\Box)$$

EXAMPLE 6.3.2. *The* $(Z_{\Box})$ *axioms are derivable in* $\mathrm{GGK}_{\Box}$ *as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,(\textsc{id})}{p, \neg p \Rightarrow}\,(\neg\Rightarrow) \quad \dfrac{\dfrac{}{p \Rightarrow p}\,(\textsc{id})}{p, \neg p \Rightarrow}\,(\neg\Rightarrow)}{p, p \Rightarrow \mid \neg p, \neg p \Rightarrow}\,(\textsc{com})}{p, p \Rightarrow \mid \neg p \Rightarrow}\,(\textsc{c})}{p \Rightarrow \mid \neg p \Rightarrow}\,(\textsc{c})}{p \Rightarrow \mid \Rightarrow \neg\neg p}\,(\Rightarrow\neg)}{\Box p \Rightarrow \mid \Rightarrow \Box\neg\neg p}\,(\Box)}{\Rightarrow \neg\Box p \mid \Rightarrow \Box\neg\neg p}\,(\Rightarrow\neg)}{\neg\neg\Box p \Rightarrow \mid \Rightarrow \Box\neg\neg p}\,(\neg\Rightarrow)}{\neg\neg\Box p \Rightarrow \mid \neg\neg\Box p \Rightarrow \Box\neg\neg p}\,(\textsc{w})}{\neg\neg\Box p \Rightarrow \Box\neg\neg p \mid \neg\neg\Box p \Rightarrow \Box\neg\neg p}\,(\textsc{w})}{\neg\neg\Box p \Rightarrow \Box\neg\neg p}\,(\textsc{ec})}{\Rightarrow \neg\neg\Box p \to \Box\neg\neg p}\,(\to\Rightarrow)$$

7 Historical remarks

For the history and development of mathematical fuzzy logic, we refer the reader to other chapters of this handbook and to the monographs [44, 41, 53]. For structural proof theory, the crucial event, both historically and scientifically, arrived in the 1930s with Gentzen's introduction of (and cut elimination proofs for) the sequent calculi LK and LJ for first-order classical logic and intuitionistic logic, respectively, to which can be traced in particular the first proofs of Theorems 2.2.2, 2.2.3, and 2.3.3 [38]. Gentzen made use of LJ and LK as tools for investigating the consistency of mathematical theories, and indeed these calculi continue to play a central role in the field of proof theory. See, e.g., the handbook [22] and monographs [64, 67] for more details.

So-called "Gentzen systems" for substructural and other non-classical logics were subsequently developed in a wide range of contexts. In linguistics, a sequent calculus was defined by Lambek in the 1950s to model the assignment of types such as "adjective" or "verb phrase" to strings of words in natural language [45]. Since the order (of words or types) as well as multiplicity is crucial in this case, sequents are (as they were for Gentzen) ordered pairs of sequences of formulas, and the calculus lacks not only weakening and contraction rules but also exchange rules for permuting formulas in sequents. A second important source of substructural logics was the family of relevance logics developed for philosophical reasons by Anderson and Belnap and co-workers from the 1960s onwards (see in particular [3, 4]), where relevance is mirrored proof-theoretically by dropping weakening rules. Conversely, logics without contraction were introduced by Grishin in the 1980s [42] with the aim of avoiding set-theoretic paradoxes involving the comprehension principle, and investigated extensively by Ono and Komori in [58]. Girard's linear logic [40], introduced in 1987 to model computational processes, drops both weakening and contraction rules but allows these to be recovered for certain formulas using special modal operators ! and ?. Comprehensive introductions to sequent calculi and other features of substructural logics and their algebraic semantics may be found in [62, 59, 37].

Hypersequents were formally introduced by Avron in 1987 as a suitable proof-theoretic framework for the "relevant-mingle" logic RM [6]; similar structures also appeared independently in a calculus defined by Pottinger for the modal logic S5 [60]. Avron subsequently introduced hypersequent calculi for three-valued Łukasiewicz logic and, crucially for the study of fuzzy logics, Gödel logic in [7]. These ideas were taken up, extended, and generalized in various directions by researchers clustered around the logic group at the Technical University of Vienna. In particular, Baaz and Zach extended Avron's calculus to first-order Gödel logic in 2000, using this system to establish the mid-hypersequent theorem and hence the Herbrand theorem for the prenex fragment of the logic [18]. These authors also provided a first syntactic elimination of the density rule, previously used by Takeuti and Titani to axiomatize first-order Gödel logic (intuitionistic fuzzy logic) in [65]. Further results on Herbrand's theorem and Skolemization in Gödel logic may be found in [9], and on the extension of Gödel logic with propositional quantifiers (in particular proofs of Theorems 6.2.1, 6.2.4, and 6.2.5) in [12, 17] (see also [16, 10]).

The proof-theoretic approach to fuzzy logics was extended to Godo and Esteva's monoidal t-norm logic MTL by Baaz, Ciabattoni, and Montagna in 2004 [11]; a hypersequent calculus is obtained by dropping contraction from the calculus for Gödel logic. Complementing this picture, hypersequent calculi were defined also for other logics with weakening, including IMTL, SMTL, $\mathrm{MTL_n}$, and $\mathrm{IMTL_n}$ ($n \geq 3$) by various authors in [25, 28, 27]. Around the same time, Metcalfe extended this picture further by introducing uninorm logic UL, obtained proof-theoretically by dropping weakening rules from the calculus for MTL, and related logics such as IUL, UML, and IUML [46] (see also [49, 36]). These works improved on Avron's original cut elimination proofs (based on a rather complicated "history method") by developing a "Schütte-Tait"-style approach, that eliminates largest cuts in derivations. An alternative "cut elimination by substitutions" method was also defined by Ciabattoni in [24] and used to obtain uniform proofs for single-conclusion hypersequent calculi. The related uniform proof of Theorem 4.1.1 may be found (in a more general context allowing alternative rules for connectives) together with example applications of cut elimination such as Theorems 4.1.2 and 4.1.3 in the monograph [53]. These approaches contributed, moreover, to a growing literature on syntactic and semantic classifications of proof calculi for which the cut rule is admissible – including, in particular, an algebraic method developed for sequent calculi by Okada and Terui in [56, 57, 66]. This latter approach has in turn led to the introduction by Ciabattoni, Galatos, and Terui in 2008 [29] of an algorithm that transforms axiom schema in certain syntactic classes into either sequent or hypersequent (depending on the class) single-conclusion rules preserving cut elimination (extended to multiple-conclusion calculi by Ciabattoni, Strassburger and Terui in [33]). As an example of the algorithm, this paper introduced a calculus for weak nilpotent minimum logic WNM. A calculus for nilpotent minimum logic NM was obtained as an extension of the hypersequent version of a calculus for constructive logic with strong negation by Metcalfe in [48].

As noted above, elimination of a hypersequent version of Takeuti and Titani's density rule was first obtained for first-order Gödel logic by Baaz and Zach in 2000 using a Gentzen-style proof that shifts applications of the rule upwards in derivations [18]. This method was extended to a wide range of hypersequent calculi by Metcalfe and Montagna in 2007 and used to establish standard completeness for propositional fuzzy logics [49]. The more elegant "density elimination by substitutions" method described in this chapter (Theorems 4.2.3 and 4.2.5), where applications of the density rule are removed by making suitable substitutions, was introduced by Ciabattoni and Metcalfe in [31] and used to provide classifications of standard complete (first-order) fuzzy logics.

Earlier "pre-hypersequent" work on proof theory for fuzzy logics focussed mostly on Gödel logic and Łukasiewicz logic. A first (rather complicated) sequent calculus for the former was defined by Sonobe in 1975 [63], and improved versions (terminating and contraction-free) were subsequently developed by Avellone, Ferrari, and Miglioli [5], and Dyckhoff [34]. The sequent calculus presented here (and Theorem 5.1.4) appears in [53], adapted from calculi defined by Avron and Konikowska in 2001 [8], while the sequent of relations approach (including a proof of Theorem 5.1.6), which applies to a wider class of "projective" logics, was developed by Baaz and Fermüller in the 1999 paper [13]. For Łukasiewicz logic, the first calculi introduced made use either of the cut rule [1, 61], extra syntax [43, 69], or a translation into finite-valued logics [2]. The elegant analytic sequent and hypersequent calculi defined above were developed by Metcalfe, Olivetti, and Gabbay in 2005 [52], alongside calculi for Meyer and Slaney's abelian logic, the logic of lattice-ordered abelian groups [54] (see also [30, 47]). Similar calculi for product logic and cancellative hoop logic were introduced by the same authors in [51]. The relationship of derivations in GŁ to (disjunctive) strategies for Giles's game (introduced by Giles in the 1970s [39]) is explored in detail by Fermuüller and Metcalfe in [35], while the uniform r-hypersequent approach for Gödel logic, Łukasiewicz logic, and product logic was developed by Ciabattoni, Fermüller, and Metcalfe in [26].

More details of the general proof-theoretic approach to first-order fuzzy logics described in Section 6.1, in particular proofs of Theorems 6.1.3, 6.1.4, 6.1.5, and 6.1.8, may be found in the monograph [53], generalizing earlier results given in the context of first-order Gödel logic [18] and monoidal t-norm logic [11]. The proof-theoretic results described here for first-order Łukasiewicz logic are taken from the papers of Baaz and Metcalfe [15, 14]. Modal fuzzy logics and the related topic of fuzzy description logics are currently the subject of intensive investigation, in particular by researchers in Barcelona (see, e.g., [20, 19]). The general proof-theoretic approach for adding modalities to hypersequent calculi that results in modal fuzzy logics complete with respect to chains is described in [32], while the particular case of fragments of Gödel modal logics (introduced in [23]) is considered in [50].

Finally, note that no Gentzen system has been presented in this chapter for Hájek's basic logic BL, the logic of continuous t-norms. Basic logic is one of the most important and widely studied fuzzy logics, and from an algebraic perspective, is rather natural: the variety of BL-algebras is generated not only by all standard (continuous t-norm based)

BL-algebras, but even by just one such algebra. In fact this algebra provides one route to defining a calculus of sorts. Information about the positioning of the valuations of the formula can be encoded by structural features of the calculus and ultimately the question of the validity of a formula in BL can be reduced (as in Łukasiewicz logic) to solving linear programming problems. This approach, developed in [55, 21], does not provide an elegant calculus for BL but does at least give a reasonable algorithm for deciding questions of validity in this logic. An alternative approach by Vetterlein [68], based on relational hypersequents, gives an elegant calculus for the product-free fragment of BL, but requires the addition of an extra modal operator in order to capture the full logic.

Acknowledgements

The author acknowledges support from Marie Curie Reintegration grant PIRG06-GA-2009-256492 and Swiss National Science Foundation grant 20002_129507, and would also like to thank the readers Agata Ciabattoni and Kazushige Terui for their very helpful comments and suggestions.

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