Admissible Rules in Logic and Algebra

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Derivability vs Admissibility

Consider a system defined by two rules:

$$Nat(0) \qquad \text{and} \qquad Nat(x) \rhd Nat(s(x)).$$

The following rule is **derivable**:

$$Nat(x) \rhd Nat(s(s(x))).$$

However, this rule is only **admissible**:

$$Nat(s(x)) \rhd Nat(x).$$

But what if we add to the system:

$$Nat(s(-1)) \qquad ???$$

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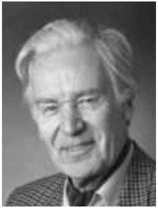
Background: Lorenzen's Idea

The notion of an **admissible rule** was introduced explicitly by Paul Lorenzen in the 1950s in the context of **intuitionistic propositional logic** IPC.

P. Lorenzen. Einführung in die operative Logik und Mathematik. Springer, 1955.



Lorenzen calls a rule R *admissible* in a system S, if adding R to the primitive rules of S does not enlarge the set of theorems.

His "operative interpretation" is that a rule R is admissible in S if every application of R can be eliminated from the extended calculus.

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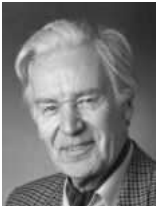

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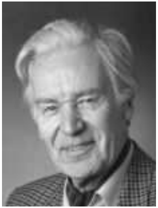

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Background: Admissibility in IPC

Examples of admissible but not derivable rules of IPC include the "independence of premises" rule

$$\{\neg p \to (q \lor r)\} \rhd (\neg p \to q) \lor (\neg p \to r)$$

and the "disjunction property"

$$\{p \lor q\} \rhd \{p, q\}.$$

It was shown by Vladimir Rybakov (among other things) that the set of admissible rules of IPC is decidable but not finitely axiomatizable.

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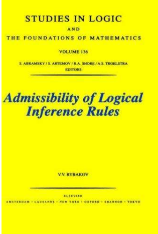

Iemhoff and Rozière proved independently that the "Visser rules"

$$\{\bigwedge_{i=1}^{n}(p_i \to q_i) \to (p_{n+1} \lor p_{n+2})\} \rhd \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_j) \quad n = 2, 3, \ldots$$

plus the disjunction property provide a "basis" for admissibility in IPC.

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Background: The Disjunctive Syllogism

In **relevant logics** such as R and RM, the "disjunctive syllogism"

$$\{\neg p,\ p \vee q\} \ \triangleright\ q$$

is admissible but not derivable.

A. R. Anderson and N. D. Belnap. *Entailment*.
Princeton University Press, 1975.

Background: Admissibility in Modal Logics

In the **modal logics** K and $\mathrm{K4}$, the rule

$$\{\Box p\} \ \triangleright \ p$$

is admissible and non-derivable, while Löb's rule

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For classes of algebraic structures, rules correspond to **clauses** (or **quasiequations**) and admissibility to validity in **free algebras**.

For example, the quasiequation

$$\{x + x \approx 0\} \rhd x \approx 0$$

- is not valid in all abelian groups (e.g., $\mathbf{Z_2}$)
- but is valid in all free abelian groups, since

$$\varphi + \varphi \approx 0 \text{ is valid in } \mathbf{Z} \quad \Longrightarrow \quad \varphi \approx 0 \text{ is valid in } \mathbf{Z}.$$

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The following clauses are valid in all **free lattices**, but not all lattices

- Whitman's condition

$$\{x \wedge y \preccurlyeq z \vee w\} \rhd \{x \wedge y \preccurlyeq z,\ x \wedge y \preccurlyeq w,\ x \preccurlyeq z \vee w,\ y \preccurlyeq z \vee w\}.$$

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- Jónsson's semi-distributivity property

$$\{x \wedge y \approx x \wedge z\} \rhd x \wedge y \approx x \wedge (y \vee z).$$

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Background: Admissibility in Proof Theory

In Gentzen's sequent calculus for propositional classical logic, applications of the **cut rule** can be *eliminated* from derivations.

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Saying that $\{\varphi_1, \ldots, \varphi_n\} \rhd \psi$ is derivable in a sequent calculus when $\frac{\Rightarrow \varphi_1 \ \ldots \ \Rightarrow \varphi_n}{\Rightarrow \psi}$ is derivable, it follows that the **transitivity rule**

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(II) An Algebraic Perspective

(III) Unification and Admissibility

(IV) Proof Theory for Admissible Rules

(V) A First-Order Framework

(VI) Eliminations and Applications.

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Part I

Admissibility in Logic

Rules

We will make use of

- **propositional languages** $\mathcal{L}$ consisting of connectives such as $\wedge, \vee, \cdot, , \to, \neg, \bot, \top$ with specified finite arities
- finite sets (denoted Γ, Δ) of $\mathcal{L}$-**formulas** (denoted ψ, φ, χ) from $\mathrm{Fm}_{\mathcal{L}}$ built from a countably infinite set of variables (denoted p, q, r)
- endomorphisms on $\mathbf{Fm}_{\mathcal{L}}$ called $\mathcal{L}$-**substitutions** (denoted σ).

Definition

An $\mathcal{L}$-**rule** is an ordered pair of finite sets of $\mathcal{L}$-formulas, written

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- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \cup \Gamma' \vdash_{\mathrm{L}} \varphi$ (monotonicity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi$, then $\Gamma \vdash_{\mathrm{L}} \psi$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ (structurality).

An L-**theorem** is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

(Note: A **finitary structural consequence relation** is obtained by fixing for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma' \vdash_{\mathrm{L}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$.)

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- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi$, then $\Gamma \vdash_{\mathrm{L}} \psi$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ (structurality).

An L-**theorem** is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

(Note: A **finitary structural consequence relation** is obtained by fixing for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma' \vdash_{\mathrm{L}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$.)

Definition

A **logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of single-conclusion $\mathcal{L}$-rules satisfying (writing $\Gamma \vdash_{\mathrm{L}} \Delta$ for $(\Gamma, \Delta) \in \mathrm{L}$):

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Multiple-Conclusion Consequence

Definition

An **m-logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of $\mathcal{L}$-rules satisfying:

- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathrm{L}} \Delta$, then $\Gamma \cup \Gamma' \vdash_{\mathrm{L}} \Delta' \cup \Delta$ (monotonicity)
- if $\Gamma \vdash_{\mathrm{L}} \{\varphi\} \cup \Delta$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \Delta$, then $\Gamma \vdash_{\mathrm{L}} \Delta$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \Delta$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\Delta$ for any $\mathcal{L}$-substitution σ (structurality).

Derivable and Admissible Rules

Definition

For a logic L on $\mathbf{Fm}_{\mathcal{L}}$, an $\mathcal{L}$-rule $\Gamma \rhd \Delta$ is

- L-**derivable**, written $\Gamma \vdash_{\mathrm{L}} \Delta$, if $\Gamma \vdash_{\mathrm{L}} \varphi$ for some $\varphi \in \Delta$.
- L-**admissible**, written $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, if for every $\mathcal{L}$-substitution σ:

$$\vdash_{\mathrm{L}} \sigma\varphi \quad \text{for all } \varphi \in \Gamma \qquad \Longrightarrow \qquad \vdash_{\mathrm{L}} \sigma\psi \quad \text{for some } \psi \in \Delta.$$

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Example: Łukasiewicz Logic

Łukasiewicz logic Ł with negation $\neg$ and implication $\to$ can be defined via an axiom system or, semantically, via a matrix with truth values $[0, 1]$, designated truth value 1, and interpretations:

$$\neg x = 1 - x \qquad \text{and} \qquad x \to y = \min(1, 1 - x + y).$$

Then for example:

$$\{p \to \neg p, \ \neg p \to p\} \not\vdash_{Ł} q, \qquad \text{but} \quad \{p \to \neg p, \ \neg p \to p\} \mathrel{|\!\sim}_{Ł} q.$$

Also, defining $\varphi \cdot \psi = \neg(\varphi \to \neg\psi)$ (so that $x \cdot y = \max(0, x + y - 1)$):

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Another Characterization of Admissibility (1)

Lemma

The following are equivalent for any logic L *on* $\mathbf{Fm}_{\mathcal{L}}$:

(1) $\Gamma \triangleright \varphi$ *is* L*-admissible.*

(2) *For the smallest logic* L^* *on* $\mathbf{Fm}_{\mathcal{L}}$ *containing* L *and* $(\Gamma, \{\varphi\})$:

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Definition

A logic L on $\mathbf{Fm}_{\mathcal{L}}$ is

- **structurally complete** if for all single-conclusion $\mathcal{L}$-rules $\Gamma \rhd \varphi$

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Classical Logic is Structurally Complete

Theorem (Pogorzelski)

Classical propositional logic CPC *is structurally complete.*

Proof.

Suppose that $\Gamma \nvdash_{\mathrm{CPC}} \varphi$. Then for some classical evaluation e, we have $e(\psi) = 1$ for all $\psi \in \Gamma$ and $e(\varphi) = 0$. Let:

$$\sigma p = \begin{cases} \top & \text{if } e(p) = 1 \\ \bot & \text{if } e(p) = 0. \end{cases}$$

It follows inductively that $\vdash_{\mathrm{CPC}} \sigma\psi \leftrightarrow \top$ iff $e(\psi) = 1$, and $\vdash_{\mathrm{CPC}} \sigma\psi \leftrightarrow \bot$ iff $e(\psi) = 1$. So $\vdash_{\mathrm{CPC}} \sigma\psi$ for all $\psi \in \Gamma$, but $\nvdash_{\mathrm{CPC}} \sigma\varphi$. I.e., $\Gamma \not\mid\!\sim_{\mathrm{CPC}} \varphi$. $\square$

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An L-**unifier** of $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is an $\mathcal{L}$-substitution σ such that

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In this case, Γ is said to be L-**unifiable**.

Notice that for a non-trivial logic L:

$$\Gamma \text{ is L-unifiable} \qquad \text{iff} \qquad \Gamma \rhd \emptyset \text{ is not L-admissible.}$$

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Exact Unifiers

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($\Rightarrow$) Let σ be an exact L-unifier of Γ. If $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, then $\vdash_{\mathrm{L}} \sigma\varphi$ for some $\varphi \in \Delta$. So $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \vdash_{\mathrm{L}} \Delta$ as required. □

Exact Unifiers

Definition

An L-unifier σ of $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is called **exact** if for all $\varphi \in \mathrm{Fm}_{\mathcal{L}}$:

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In this case, Γ is said to be L-**exact**.

Lemma

If Γ is L-exact, then $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$ iff $\Gamma \vdash_{\mathrm{L}} \Delta$.

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Suppose that L has an implication $\to$ satisfying $\{p, p \to q\} \vdash_{\mathrm{L}} q$.

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Notice that every L-projective unifier of $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is

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- a **most general** L**-unifier** of Γ
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A logic L on $\mathbf{Fm}_{\mathcal{L}}$ is called

- **hereditarily structurally complete** if each logic on $\mathbf{Fm}_{\mathcal{L}}$ extending L is structurally complete.
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In particular, if *all* finite $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ are

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For each **implication-conjunction** formula φ of IPC, define

$$\sigma(p) = \varphi \to p \quad \text{for each variable } p.$$

Then inductively, for each implication-conjunction formula ψ:

$$\vdash_{\mathrm{IPC}} \sigma\psi \to (\varphi \to \psi) \quad \text{and} \quad \vdash_{\mathrm{IPC}} (\varphi \to \psi) \to \sigma\psi.$$

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The $\{\to, \land\}$ fragment of IPC is hereditarily universally complete.

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In fact, the $\{\to\}$, $\{\to,\land\}$, and $\{\to,\land,\neg\}$ fragments of *all* **intermediate logics** are hereditarily universally complete.

However, this is not the case for the $\{\to,\neg\}$ fragments; e.g.

$$\{p \to \neg q,\ (\neg\neg p \to p) \to r,\ (\neg\neg q \to q) \to r\} \rhd r$$

is admissible but not derivable in the $\{\to,\neg\}$ fragment of IPC.

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Part II

An Algebraic Perspective

The Idea

For classes of algebraic structures:

- rules correspond to **clauses**
- single-conclusion rules correspond to **quasiequations**
- logics correspond to **quasivarieties**
- admissibility corresponds to validity in **free algebras**.

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We fix an **algebraic language** $\mathcal{L}$ and consider classes of $\mathcal{L}$**-algebras**.

$\mathbf{Fm}_{\mathcal{L}}(X)$ denotes the **formula algebra of** $\mathcal{L}$ over a set of variables X, writing just $\mathbf{Fm}_{\mathcal{L}}$ when X is countably infinite.

An $\mathcal{L}$**-equation** is an ordered pair of $\mathcal{L}$-formulas, written $\varphi \approx \psi$.

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Validity

An $\mathcal{L}$-clause $\Gamma \rhd \Delta$ is **valid** in an $\mathcal{L}$-algebra **A**, written

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if for every homomorphism $g \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$,

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A K-**unifier** of a set of $\mathcal{L}$-equations Γ is a homomorphism $\sigma\colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ such that:

$$\models_{\mathsf{K}} \sigma(\varphi) \approx \sigma(\psi) \quad \text{for all } \varphi \approx \psi \in \Gamma.$$

An $\mathcal{L}$-**clause** $\Gamma \rhd \Delta$ is K-**admissible** if

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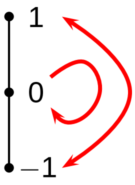
Example: A Three Element Algebra

Consider the Kleene lattice $\mathbf{C_3} = \langle \{-1, 0, 1\}, \wedge, \vee, \neg \rangle$ described by:



Then (since no formula is constantly 0)

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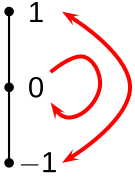

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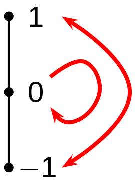

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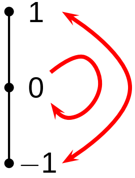

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The Congruence $\sim_K$

For a class of $\mathcal{L}$-algebras K and $\mathrm{Fm}_{\mathcal{L}}(X) \neq \emptyset$, define for $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}(X)$:

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Then $\sim_K$ is a **congruence** on $\mathbf{Fm}_{\mathcal{L}}(X)$, i.e., an equivalence relation satisfying for each n-ary function symbol f of $\mathcal{L}$:

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Free Algebras

The *quotient algebra*

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with universe $\mathrm{F}_{\mathsf{K}}(X) = \{[\varphi]_{\sim_{\mathsf{K}}} \mid \varphi \in \mathrm{Fm}_{\mathcal{L}}(X)\}$ and operations

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Lemma

$\Gamma \rhd \Delta$ *is* K*-admissible iff* $\Gamma \models_{\mathbf{F}_{\mathsf{K}}(\omega)} \Delta$.

Proof.

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Any map sending each variable p to a member of the equivalence class $g(p)$ extends to a homomorphism $\sigma \colon \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$.

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Varieties are also (Birkhoff's theorem) the classes of $\mathcal{L}$-algebras closed under taking **homomorphic images**, **subalgebras**, and **products**.

$\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K})$ is the **smallest variety containing** K, and

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Admissibility Algebraically

Theorem

The following are equivalent:

(i) $\Gamma \rhd \varphi \approx \psi$ *is* K*-admissible*

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If K-admissibility and K-validity coincide for $\mathcal{L}$*-quasiequations*, that is,

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then K is called **structurally complete**.

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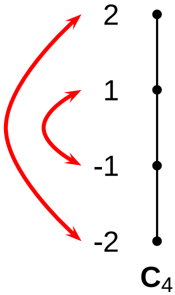

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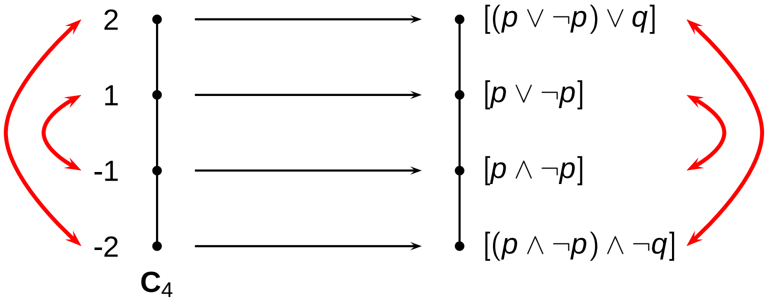

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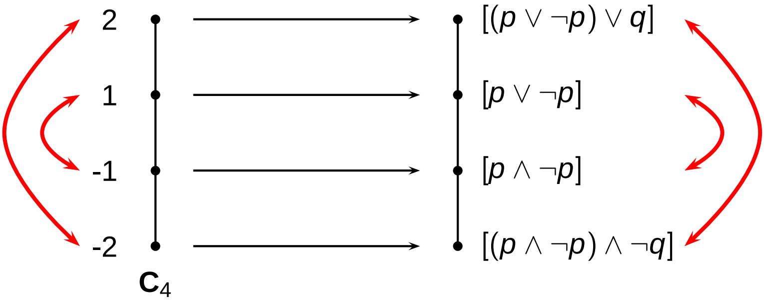

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Checking Admissibility in Finite Algebras

For a **finite** $\mathcal{L}$-algebra **A**:

$$\Gamma \rhd \varphi \approx \psi \text{ is } \mathbf{A}\text{-admissible} \qquad \text{iff} \qquad \Gamma \models_{\mathbf{F_A}(|A|)} \varphi \approx \psi.$$

Also, $\mathbf{F_A}(|A|)$ is finite, so checking **A**-admissibility is **decidable**.

But $\mathbf{F_A}(n)$ can be big even for small $|A|$ and n, e.g., $|\mathbf{F}_{\mathbf{C_3}}(2)| = 82\ldots$

However, for any (perhaps small!) *subalgebra* **B** of $\mathbf{F_A}(|A|)$ such that **A** is a *homomorphic image* of **B**:

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For a finite algebra **A**:

(i) Compute the set $\mathbb{S}(\mathbf{F_A}(|A|))$ of subalgebras of $\mathbf{F_A}(|A|)$.

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Steps (i)-(ii) have been implemented using the Algebra Workbench; step (iii) can be implemented using, e.g., MUltlog/MUltseq or $_3T^AP$.

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$\to$	-1	0	1
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The procedure discovers a subalgebra of the 60-element free algebra $\mathbf{F}_{\mathbf{S_3^{\to}}}(2)$ isomorphic to $\mathbf{S_3^{\to}}$, and hence that $\mathbf{S_3^{\to}}$ is structurally complete.

Structural completeness has also been confirmed for the 3-element implicational Łukasiewicz algebra, Gödel algebra, and Stone algebra.

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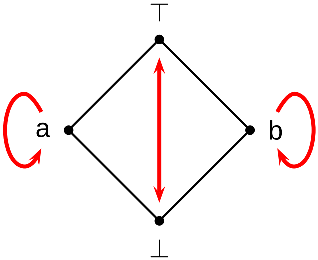

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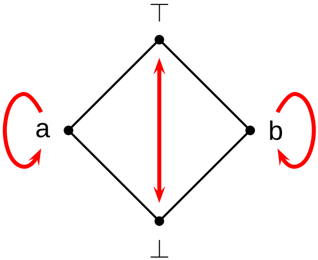

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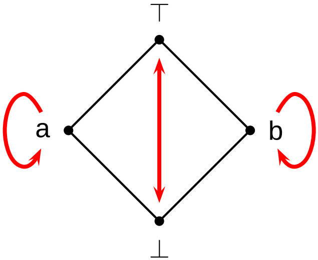

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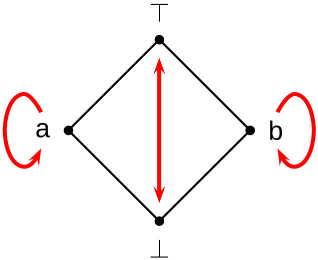

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Other Examples

- For the De Morgan algebra

$$\mathbf{D_4^b} = \langle \{\bot, \mathrm{a}, \mathrm{b}, \top\}, \wedge, \vee, \neg, \bot, \top \rangle$$

the procedure finds a smallest 10-element algebra in $\mathrm{Adm}(\mathbf{D_4})$.

- For the Kleene lattice and Kleene algebra

$$\mathbf{C_3} = \langle \{\top, \mathrm{a}, \bot\}, \wedge, \vee, \neg \rangle \quad \text{and} \quad \mathbf{C_3^b} = \langle \{\top, \mathrm{a}, \bot\}, \wedge, \vee, \neg, \bot, \top \rangle$$

the procedure finds smallest 4-element chains.

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- For the De Morgan algebra

$$\mathbf{D_4^b} = \langle \{\bot, \mathrm{a}, \mathrm{b}, \top\}, \wedge, \vee, \neg, \bot, \top \rangle$$

  the procedure finds a smallest 10-element algebra in $\mathrm{Adm}(\mathbf{D_4})$.

- For the Kleene lattice and Kleene algebra

$$\mathbf{C_3} = \langle \{\top, \mathrm{a}, \bot\}, \wedge, \vee, \neg \rangle \quad \text{and} \quad \mathbf{C_3^b} = \langle \{\top, \mathrm{a}, \bot\}, \wedge, \vee, \neg, \bot, \top \rangle$$

  the procedure finds smallest 4-element chains.

Experiments in Admissibility

$\mathbf{A}$	$\lvert\mathbf{A}\rvert$	Quasivariety $\mathbb{Q}(\mathbf{A})$	Free algebra	$\lvert$Output Algebra$\rvert$
$\mathbf{Ł_3}$	3	algebras for Ł_3	$\lvert\mathbf{F_A}(1)\rvert = 12$	6
$\mathbf{Ł_3^{\to}}$	3	algebras for $\text{Ł}_3^{\to}$	$\lvert\mathbf{F_A}(2)\rvert = 40$	3
$\mathbf{B_1}$	3	Stone algebras	$\lvert\mathbf{F_A}(1)\rvert = 6$	3
$\mathbf{C_3^b}$	3	Kleene algebras	$\lvert\mathbf{F_A}(1)\rvert = 6$	4
$\mathbf{C_3}$	3	Kleene lattices	$\lvert\mathbf{F_A}(2)\rvert = 82$	4
$\mathbf{S_3^{\to\neg}}$	3	algebras for $\mathrm{RM}^{\to\neg}$	$\lvert\mathbf{F_A}(2)\rvert = 264$	6
$\mathbf{S_3^{\to}}$	3	algebras for $\mathrm{RM}^{\to}$	$\lvert\mathbf{F_A}(2)\rvert = 60$	3
$\mathbf{G_3}$	3	algebras for G_3	$\lvert\mathbf{F_A}(2)\rvert = 18$	3
$\mathbf{D_4}$	4	De Morgan lattices	$\lvert\mathbf{F_A}(2)\rvert = 166$	8
$\mathbf{D_4^b}$	4	De Morgan algebras	$\lvert\mathbf{F_A}(2)\rvert = 168$	10
$\mathbf{S_4^{\to\neg\mathrm{e}}}$	4	$\mathbb{Q}(\mathbf{S_4^{\to\neg\mathrm{e}}})$	$\lvert\mathbf{F_A}(1)\rvert = 18$	6
$\mathbf{B_2}$	5	$\mathbb{Q}(\mathbf{B_2})$	$\lvert\mathbf{F_A}(1)\rvert = 7$	5

A Last Technical Comment

Let $\mathcal{Q}$ be a $\mathcal{L}$-quasivariety.

- $\mathbf{A} \in \mathcal{Q}$ is called **exact** if it embeds into a free algebra of $\mathcal{Q}$.
- Given a finite set of $\mathcal{L}$-equations Γ over a finite set of variables X, define a congruence on $\mathbf{F}_{\mathcal{Q}}(X)$ by

$$[\varphi] \sim_\Gamma [\psi] \quad \text{iff} \quad \Gamma \models_{\mathcal{Q}} \varphi \approx \psi.$$

- If the **finitely presented algebra** $\mathbf{F}_{\mathcal{Q}}(X) \,/ \sim_\Gamma$ is exact, then

$$\Gamma \models_{\mathcal{Q}} \Delta \quad \text{iff} \quad \Gamma \mathrel{|\!\sim}_{\mathcal{Q}} \Delta.$$

- $\mathbf{A} \in \mathcal{Q}$ is **projective** if it is the *retract* of a free algebra of $\mathcal{Q}$ (i.e., embeds into and is the homomorphic image of a free algebra).

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Part III

Unification and Admissibility

The Idea

Admissibility is a more general problem than unification, but in certain cases we can “reduce” admissibility questions to unification questions.

For convenience, we give a “logical” account of unification.

In particular, we consider a **modal** or **intermediate logic** L and can therefore treat formulas rather than finite sets of formulas.

S. Ghilardi. Unification in intuitionistic logic.
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L-Unification

Given substitutions σ_1, σ_2:

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Recall that a substitution σ is an L-**unifier** of a formula φ if $\vdash_{\mathrm{L}} \sigma(\varphi)$.

A set $\mathcal{C}$ of L-unifiers of φ is called **complete** if

- for any L-unifier σ of φ, there exists $\sigma' \in \mathcal{C}$ such that $\sigma' \leq_{\mathrm{L}} \sigma$.

$\mathcal{C}$ is **minimal** if also

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If $\mathcal{C} = \{\sigma\}$, then σ is a **most general unifier** of φ.

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Unification Type

If there exists a minimal complete set of L-unifiers $\mathcal{C}$ for φ

- with $|\mathcal{C}| = 1$, then φ has type **unitary**
- with $|\mathcal{C}|$ finite and $|\mathcal{C}| \neq 1$, then φ has type **finitary**
- with $|\mathcal{C}|$ infinite, then φ has type **infinitary**.

Otherwise, φ has type **nullary**.

The **unification type** of L is the maximal type of φ as ranked by

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E.g., CPC unitary; IPC finitary (Rozière 1995); $\mathrm{K4}$ finitary (Ghilardi 2000); K nullary (Jeřábek 2011); Ł nullary (Marra & Spada 2011).

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Recall that a rule $\{\varphi\} \rhd \Delta$ is L**-admissible**, if every L-unifier of φ is an L-unifier of some formula in Δ.

On the one hand:

$$\varphi \text{ is L-unifiable} \quad \Longleftrightarrow \quad \{\varphi\} \rhd \emptyset \ \text{ is not L-admissible.}$$

But also, if $\mathcal{C}$ is a **complete set of** L**-unifiers** for φ, then

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So if L is **decidable** and **finitary** and we can effectively find finite complete sets of L-unifiers, then L-admissibility is decidable.

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Definition

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$$\{\varphi\} \vdash_{\mathrm{L}} \sigma\psi \leftrightarrow \psi \qquad \text{for all } \psi \in \mathrm{Fm}_{\mathcal{L}}.$$

In this case, φ is said to be L-**projective**.

Lemma

If φ is L-projective, then $\{\varphi\} \mathrel{\vert\!\sim}_{\mathrm{L}} \Delta$ iff $\{\varphi\} \vdash_{\mathrm{L}} \Delta$.

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A class of Kripke models $\mathcal{K}$ has the **extension property** if given $K_1, \ldots, K_n \in \mathcal{K}$, there is a Kripke model in $\mathcal{K}$ obtained by attaching one new node below all nodes in $K_1, \ldots, K_n$.

Theorem (Ghilardi 1999)

A formula is IPC*-projective iff its class of finite Kripke models has the extension property.*

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An IPC*-projective approximation can be found effectively for any formula.*

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Axiomatizing Admissibility

For a logic L, we are interested in finding a set of rules that "axiomatizes" (over L) the admissible rules of L.

Definition

A **basis** for $\mathrel{|\!\sim}_{\mathrm{L}}$ over L is a set B of rules such that $\mathrel{|\!\sim}_{\mathrm{L}}$ is the smallest m-logic extending $B \cup \mathrm{L}$.

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Iemhoff and Rozière proved independently that the "Visser rules"

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plus the disjunction property provide a basis for admissibility in IPC.

P. Rozière. Regles admissibles en calcul propositionnel intuitionniste. Ph.D. thesis, Université Paris VII, 1992.

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A Last Example

The following "Wroński rules" ($n \in \mathbb{N}$):

$$(\mathrm{W}_n) \quad \{p_1 \to \ldots \to p_{n-1} \to \neg p_n\} \rhd \{\neg\neg p_1 \to p_1, \ldots, \neg\neg p_n \to p_n\}$$

provide a basis for the admissible rules of the implication-negation fragment of any intermediate logic.

P. Cintula and G. Metcalfe. Admissible rules in the implication-negation fragment of intuitionistic logic. *Annals of Pure and Applied Logic* 162(2): 162–171 (2010).

Part IV

Proof Theory for Admissible Rules

The Idea

Recall that a rule $\Gamma \rhd \Delta$ is L**-admissible** if every L-unifier of Γ is an L-unifier of some formula in Δ.

Since the sets of admissible rules of IPC and extensible modal logics are **recursively enumerable**, they are also **decidable**.

However, we would also like to have analytic "Gentzen-style" calculi for deciding the admissibility of rules in these logics.

Instead of treating sequents as the proof objects of a calculus, we deal with **sequent rules**.

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$$\Gamma \Rightarrow \Delta,$$

and interpreted by (with $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$)

$$\mathcal{I}(S) = \bigwedge \Gamma \to \bigvee \Delta.$$

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- IPC-**derivable** if $\{\mathcal{I}(S) \mid S \in \mathcal{G}\} \vdash_{\mathrm{IPC}} \{\mathcal{I}(S) \mid S \in \mathcal{H}\}$
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For example, the sequent rule

$$\{(\neg p \Rightarrow q, r)\} \rhd \{(\neg p \Rightarrow q), (\neg p \Rightarrow r)\}$$

is IPC-admissible, since

$$\{\neg p \to (q \lor r)\} \mathrel{|\!\sim}_{\mathrm{IPC}} \{\neg p \to q, \neg p \to r\}.$$

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Our **proof system** GAMI for admissibility in IPC treats *sequent rules* $\mathcal{G} \triangleright \mathcal{H}$ as proof objects, and consists of

- right rules
- left rules
- structural rules
- a Visser rule.

Right Rules

$$\frac{}{\mathcal{G} \rhd (\Gamma, \varphi \Rightarrow \varphi, \Delta), \mathcal{H}} \text{ (ID)}$$

$$\frac{}{\mathcal{G} \rhd (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}} \ \rhd(\bot\Rightarrow) \qquad \frac{\mathcal{G} \rhd (\Gamma \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow \bot, \Delta), \mathcal{H}} \ \rhd(\Rightarrow\bot)$$

$$\frac{\mathcal{G} \rhd (\Gamma \Rightarrow \varphi, \Delta), \mathcal{H} \quad \mathcal{G} \rhd (\Gamma \Rightarrow \psi, \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow \varphi \wedge \psi, \Delta), \mathcal{H}} \ \rhd(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \rhd (\Gamma, \varphi, \psi \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma, \varphi \wedge \psi \Rightarrow \Delta), \mathcal{H}} \ \rhd(\wedge\Rightarrow)$$

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$$\frac{\mathcal{G} \rhd (\Gamma \Rightarrow \varphi, \Delta), \mathcal{H} \quad \mathcal{G} \rhd (\Gamma, \psi \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma, \varphi \to \psi \Rightarrow \Delta), \mathcal{H}} \ \rhd(\to\Rightarrow) \qquad \frac{\mathcal{G} \rhd (\Gamma, \varphi \Rightarrow \psi), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow \varphi \to \psi, \Delta), \mathcal{H}} \ \rhd(\Rightarrow\to)$$

Left Rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \rhd \mathcal{H}} \ (\bot\Rightarrow)_\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \rhd \mathcal{H}} \ (\Rightarrow\bot)_\rhd$$

$$\frac{\mathcal{G}, (\Gamma, \varphi, \psi \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \varphi \wedge \psi \Rightarrow \Delta) \rhd \mathcal{H}} \ (\wedge\Rightarrow)_\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \varphi, \Delta), (\Gamma \Rightarrow \psi, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \varphi \wedge \psi, \Delta) \rhd \mathcal{H}} \ (\Rightarrow\wedge)_\rhd$$

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$$\frac{\mathcal{G}, (\Gamma, \psi \Rightarrow \Delta), (\Gamma, \varphi \to \psi \Rightarrow \varphi, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \varphi \to \psi \Rightarrow \Delta) \rhd \mathcal{H}} \ (\to)_\rhd$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, \varphi \Rightarrow \psi) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \varphi \to \psi, \Delta) \rhd \mathcal{H}} \ (\Rightarrow\to)_\rhd \qquad p \text{ new}$$

$$\frac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow \varphi), (\psi \Rightarrow q) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \varphi \to \psi \Rightarrow \Delta) \rhd \mathcal{H}} \ (\to\Rightarrow)_\rhd \qquad p, q \text{ new}$$

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$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \rhd \mathcal{H}} \;(\bot\Rightarrow)_\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \rhd \mathcal{H}} \;(\Rightarrow\bot)_\rhd$$

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$$\frac{\mathcal{G} \triangleright \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \triangleright \mathcal{H}} \ (\bot\Rightarrow)_\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \triangleright \mathcal{H}} \ (\Rightarrow\bot)_\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, \varphi, \psi \Rightarrow \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (\Gamma, \varphi \land \psi \Rightarrow \Delta) \triangleright \mathcal{H}} \ (\land\Rightarrow)_\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \varphi, \Delta), (\Gamma \Rightarrow \psi, \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \varphi \land \psi, \Delta) \triangleright \mathcal{H}} \ (\Rightarrow\land)_\triangleright$$

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Structural Rules

Weakening Rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \ (\mathrm{W})\rhd \qquad \frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \ \rhd(\mathrm{W})$$

Anti-Cut Rule

$$\frac{\mathcal{G}, (\Gamma, \Gamma' \Rightarrow \Delta', \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \varphi \Rightarrow \Delta), (\Gamma' \Rightarrow \varphi, \Delta') \rhd \mathcal{H}} \ (\mathrm{AC})$$

Projection Rule

$$\frac{\mathcal{G} \rhd (\Gamma, \mathcal{I}(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \ (\mathrm{PJ})$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

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Examples

The following "identity" sequent rules are derivable using (PJ):

$$\frac{}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \;\triangleright\; (\Gamma, \Gamma' \Rightarrow \Delta', \Delta), \mathcal{H}} \;(\text{SID})$$

We can also derive sequent rules corresponding to the usual cut rule:

$$\frac{\dfrac{}{(\Gamma, \Gamma' \Rightarrow \Delta', \Delta) \;\triangleright\; (\Gamma, \Gamma' \Rightarrow \Delta', \Delta)} \;(\text{SID})}{(\Gamma, \varphi \Rightarrow \Delta), (\Gamma' \Rightarrow \varphi, \Delta') \;\triangleright\; (\Gamma, \Gamma' \Rightarrow \Delta', \Delta)} \;(\text{AC})$$

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A Visser Rule

$$\frac{\{\mathcal{G}, (\Gamma \Rightarrow \varphi) \rhd \mathcal{H}\}_{\varphi \in \Delta} \quad \{\mathcal{G} \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}\}_{\emptyset \neq \Pi \subseteq \Gamma_{\Delta}}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd \mathcal{H}} \ \text{(V)}$$

where Γ contains only implications, and

1. $\Gamma^{\Pi} = \{\varphi \to \psi \in \Gamma \mid \varphi \notin \Pi\}$
2. $\Gamma_{\Delta} = \{\varphi \notin \Delta \mid \exists \psi\, (\varphi \to \psi) \in \Gamma\}$.

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Example

Let (V)′ or (PJ)′ denote (V) or (PJ) with applications of (W)$\triangleright$ and $\triangleright$(W):

The two rightmost leaves in this proof tree are instances of (SID), while the derivability of the other leaf follows from the right rules.

Example

Let $(\text{V})'$ or $(\text{PJ})'$ denote (V) or (PJ) with applications of $(\text{W})\triangleright$ and $\triangleright(\text{W})$:

$$\dfrac{}{(\neg p \Rightarrow q, r) \;\triangleright\; (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\;(\rightarrow)\triangleright$$

The two rightmost leaves in this proof tree are instances of (SID), while the derivability of the other leaf follows from the right rules.

Example

Let $(\textsc{v})'$ or $(\textsc{pj})'$ denote $(\textsc{v})$ or $(\textsc{pj})$ with applications of $(\textsc{w})\triangleright$ and $\triangleright(\textsc{w})$:

$$\dfrac{\dfrac{}{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r) \ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\textsc{v})'}{(\neg p \Rightarrow q, r) \ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\to)\triangleright$$

The two rightmost leaves in this proof tree are instances of $(\textsc{sid})$, while the derivability of the other leaf follows from the right rules.

Example

Let $(\textsc{v})'$ or $(\textsc{pj})'$ denote $(\textsc{v})$ or $(\textsc{pj})$ with applications of $(\textsc{w})\triangleright$ and $\triangleright(\textsc{w})$:

$$\dfrac{\dfrac{}{(\neg p \Rightarrow p) \;\triangleright\; (\neg p \Rightarrow q)}\;(\textsc{pj})' \qquad \dfrac{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r) \;\triangleright\; (\neg p \Rightarrow q), (\neg p \Rightarrow r)}{(\neg p \Rightarrow q, r) \;\triangleright\; (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\;(\to)\triangleright}{}\;(\textsc{v})'$$

The two rightmost leaves in this proof tree are instances of $(\textsc{sid})$, while the derivability of the other leaf follows from the right rules.

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Let $(\textsc{v})'$ or $(\textsc{pj})'$ denote $(\textsc{v})$ or $(\textsc{pj})$ with applications of $(\textsc{w})\triangleright$ and $\triangleright(\textsc{w})$:

$$\dfrac{\dfrac{\triangleright\ (\neg p, \neg p \to p \Rightarrow q)}{(\neg p \Rightarrow p)\ \triangleright\ (\neg p \Rightarrow q)}\ (\textsc{pj})' \qquad \dfrac{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}{(\neg p \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\to)\triangleright}{}\ (\textsc{v})'$$

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Let $(\textsc{v})'$ or $(\textsc{pj})'$ denote $(\textsc{v})$ or $(\textsc{pj})$ with applications of $(\textsc{w})\triangleright$ and $\triangleright(\textsc{w})$:

$$
\dfrac{\dfrac{\triangleright\ (\neg p, \neg p \to p \Rightarrow q)}{(\neg p \Rightarrow p)\ \triangleright\ (\neg p \Rightarrow q)}\ (\textsc{pj})' \qquad (\neg p \Rightarrow q)\ \triangleright\ (\neg p \Rightarrow q)}{\dfrac{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}{(\neg p \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\to)\triangleright}\ (\textsc{v})'
$$

The two rightmost leaves in this proof tree are instances of $(\textsc{sid})$, while the derivability of the other leaf follows from the right rules.

Example

Let $(\textsc{v})'$ or $(\textsc{pj})'$ denote $(\textsc{v})$ or $(\textsc{pj})$ with applications of $(\textsc{w})\triangleright$ and $\triangleright(\textsc{w})$:

$$\dfrac{\dfrac{\dfrac{\triangleright\ (\neg p, \neg p \to p \Rightarrow q)}{(\neg p \Rightarrow p)\ \triangleright\ (\neg p \Rightarrow q)}\ (\textsc{pj})' \qquad (\neg p \Rightarrow q)\ \triangleright\ (\neg p \Rightarrow q) \qquad (\neg p \Rightarrow r)\ \triangleright\ (\neg p \Rightarrow r)}{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\textsc{v})'}{(\neg p \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\to)\triangleright$$

The two rightmost leaves in this proof tree are instances of $(\textsc{sid})$, while the derivability of the other leaf follows from the right rules.

Example

Let $(\mathrm{V})'$ or $(\mathrm{PJ})'$ denote (V) or (PJ) with applications of $(\mathrm{W})\triangleright$ and $\triangleright(\mathrm{W})$:

$$\dfrac{\dfrac{\triangleright\ (\neg p, \neg p \to p \Rightarrow q)}{(\neg p \Rightarrow p)\ \triangleright\ (\neg p \Rightarrow q)}\ (\mathrm{PJ})' \quad (\neg p \Rightarrow q)\ \triangleright\ (\neg p \Rightarrow q) \quad (\neg p \Rightarrow r)\ \triangleright\ (\neg p \Rightarrow r)}{\dfrac{(\neg p \Rightarrow p, q, r), (\bot \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}{(\neg p \Rightarrow q, r)\ \triangleright\ (\neg p \Rightarrow q), (\neg p \Rightarrow r)}\ (\to)\triangleright}\ (\mathrm{V})'$$

The two rightmost leaves in this proof tree are instances of (SID), while the derivability of the other leaf follows from the right rules.

We also have systems for various classes of transitive modal logics;
e.g., for the Gödel-Löb logic GL, we make use of the rules:

$$\frac{\mathcal{G} \triangleright (\Box\Gamma, \Gamma, \Box\varphi \Rightarrow \varphi), \mathcal{H}}{\mathcal{G} \triangleright (\Box\Gamma, \Gamma' \Rightarrow \Box\varphi, \Delta), \mathcal{H}}$$

$$\frac{\{\mathcal{G}, (\Box\Gamma, \Gamma \Rightarrow \varphi) \triangleright \mathcal{H}\}_{\varphi \in \Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \triangleright \mathcal{H}}$$

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Modal Logics

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$$\frac{\mathcal{G} \ \triangleright\ (\Box\Gamma, \Gamma, \Box\varphi \Rightarrow \varphi), \mathcal{H}}{\mathcal{G} \ \triangleright\ (\Box\Gamma, \Gamma' \Rightarrow \Box\varphi, \Delta), \mathcal{H}}$$

$$\frac{\{\mathcal{G}, (\Box\Gamma, \Gamma \Rightarrow \varphi) \ \triangleright\ \mathcal{H}\}_{\varphi \in \Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \ \triangleright\ \mathcal{H}}$$

We also have systems for various classes of transitive modal logics; e.g., for the Gödel-Löb logic GL, we make use of the rules:

$$\frac{\mathcal{G} \;\triangleright\; (\Box\Gamma, \Gamma, \Box\varphi \Rightarrow \varphi), \mathcal{H}}{\mathcal{G} \;\triangleright\; (\Box\Gamma, \Gamma' \Rightarrow \Box\varphi, \Delta), \mathcal{H}}$$

$$\frac{\{\mathcal{G}, (\Box\Gamma, \Gamma \Rightarrow \varphi) \;\triangleright\; \mathcal{H}\}_{\varphi \in \Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \;\triangleright\; \mathcal{H}}$$

Related Work

Tableaux methods for checking admissibility in IPC and modal logics have also been developed.

S. Ghilardi. A resolution/tableaux algorithm for projective approximations in IPC. *Logic journal of the IGPL* 10(3):227–241, 2002.

S. Babenyshev, V. Rybakov, R. A. Schmidt, and D. Tishkovsky. A tableau method for checking rule admissibility in S4. *Proceedings of UNIF 2009*, ENTCS 262:17–32, 2010.

Recall also that proof systems for checking admissibility in finite-valued logics can be automatically generated:

G. Metcalfe and C. Röthlisberger. Unifiability and admissibility in finite algebras. *Proceedings of CiE 2012*, LNCS 7318: 485–495. Springer, 2012.

Complexity Results

Jeřábek has characterized the computational complexity of admissibility in various families of intermediate and modal logics.

In particular, deciding admissibility is coNEXP-complete for IPC, KC, $\mathrm{K4}$, $\mathrm{S4}$, GL, etc.

E. Jeřábek. Complexity of admissible rules.
Archive for Mathematical Logic 46(2):73–92, 2007.

A Question

Can admissible rules be *useful* for proof theory? E.g., for shortening proofs or speeding up proof search?

This is the case for the **cut rule** in sequent calculi. . . .

Note, however, that for IPC and extensible modal logics, systems with admissible rules are polynomially simulated by the original systems.

G. Mints and A. Kojevnikov. Intuitionistic Frege systems are polynomially equivalent. *Zapisky Nauchnych Seminarov POMI* 316:129–146, 2004.

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Part V

A First-Order Framework

Two Notions of Admissibility

Informally, for a system S and rules consisting of a finite sets of premises and conclusions, there are two notions of admissibility:

(A) *"A rule is* **admissible** *in S if the set of theorems of S does not change when the rule is added to the existing rules of S."*

(B) *"A rule is* **admissible** *in S if any substitution mapping all of its premises to theorems of S, also maps one of its conclusions to a theorem of S."*

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We have seen that these notions coincide for the single-conclusion rules of a logic, but not always in other cases...

The Disjunction Property

The **disjunction property**

$$\{p \lor q\} \ \rhd \ \{p, \ q\}$$

is admissible in IPC according to both (A) and (B).

The Linearity Property

However, the **linearity property**

$$\rhd \{p \to q,\ q \to p\}$$

is admissible in **Gödel logic** (i.e., $\mathrm{IPC} + (\varphi \to \psi) \vee (\psi \to \varphi)$) according to (A), but not (B).

The Density Rule

Moreover, the **density rule**

$$\{((\varphi \to p) \lor (p \to \psi)) \lor \chi\} \rhd \{(\varphi \to \psi) \lor \chi\}$$
$$\text{where } p \text{ does not occur in } \varphi, \psi, \text{ or } \chi$$

is admissible in Gödel logic according to (A), but admissibility according to (B) does not really make much sense…

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More Generally...

What does it mean for a **first-order sentence** such as

$$(\exists x)(\forall y)(x \leq y) \qquad \text{or} \qquad (\forall x)(\exists y)\neg(x \leq y)$$

to be admissible in a logic / class of algebras?

First-Order Logic

We assume the usual terminology of **first-order logic with equality**, making use of the symbols $\forall$, $\exists$, $\sqcap$, $\sqcup$, $\Rightarrow$, $\sim$, 0, 1, and $\approx$.

In particular, for a first-order language $\mathcal{L}$, Sen($\mathcal{L}$) is the set of sentences of $\mathcal{L}$ with respect to a countably infinite set of variables.

We will denote $\mathcal{L}$**-terms** by s, t, u, **(first-order)** $\mathcal{L}$**-formulas** by φ, ψ, and **sets of** $\mathcal{L}$**-formulas** by Σ, Θ.

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Preserving First-Order Sentences

Definition

For a class of $\mathcal{L}$-structures K and $\Sigma \subseteq \mathrm{Sen}(\mathcal{L})$, we set

$$\mathrm{Th}_\Sigma(\mathsf{K}) = \{\psi \in \Sigma \mid \mathsf{K} \models \psi\}$$

and say that $\varphi \in \mathrm{Sen}(\mathcal{L})$ **preserves** Σ **in** K if

$$\mathrm{Th}_\Sigma(\mathsf{K}) = \mathrm{Th}_\Sigma(\{\mathbf{A} \in \mathsf{K} \mid \mathbf{A} \models \varphi\}).$$

If K is axiomatized by $\Theta \subseteq \mathrm{Sen}(\mathcal{L})$, then φ *preserves* Σ *in* K when:

$$\text{For all } \psi \in \Sigma: \quad \Theta \models \psi \quad \text{iff} \quad \Theta \cup \{\varphi\} \models \psi.$$

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Consider the variety BA of **Boolean algebras** in a language $\mathcal{L}_{\text{Bool}}$ and

$$\varphi = (\forall x)((x \approx \bot) \sqcup (x \approx \top)).$$

Then φ preserves the set of $\mathcal{L}_{\text{Bool}}$-equations in BA, but $\mathbf{F}_{\text{BA}}(\omega) \not\models \varphi$.

Note that $\neg\varphi$ also preserves the set of $\mathcal{L}_{\text{Bool}}$-equations in BA.

Consider the variety BA of **Boolean algebras** in a language $\mathcal{L}_{\mathrm{Bool}}$ and

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Example: Boolean Algebras

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Theorem

The following are equivalent for an $\mathcal{L}$**-quasivariety** $\mathcal{Q}$ *and* $\mathcal{L}$**-quasiequation** φ*:*

(i) φ *is* $\mathcal{Q}$*-admissible*

(ii) $\mathbf{F}_{\mathcal{Q}}(\omega) \models \varphi$

(iii) $\mathbb{V}(\mathcal{Q}) = \mathbb{V}(\{\mathbf{A} \in \mathcal{Q} \mid \mathbf{A} \models \varphi\})$

(iv) φ *preserves the set of* $\mathcal{L}$**-equations** *in* $\mathcal{Q}$.

Quasivarieties and Quasiequations

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*The following are equivalent for an $\mathcal{L}$-**quasivariety** $\mathcal{Q}$ and $\mathcal{L}$-**clause** φ:*

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Positive Clauses Preserving Equations

Theorem

If $\mathcal{V}$ is a **congruence distributive $\mathcal{L}$-variety**, *then the following are equivalent for any positive $\mathcal{L}$-clause φ:*

(i) $\mathbf{A} \models \varphi$ *for all* **subdirectly irreducible** *algebras* $\mathbf{A} \in \mathcal{V}$

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For example, the positive clause

$$\rhd \{x \leq y, \; y \leq x\}.$$

is valid in all subdirectly irreducible Gödel Algebras.

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Skolemization

Consider an algebraic language $\mathcal{L}$ and a **prenex formula** $\varphi \in \mathrm{Sen}(\mathcal{L})$.

The **Skolem form** $\mathrm{sk}(\varphi) \in \mathrm{Sen}(\mathcal{L}')$ of φ is obtained by repeating

$$(\forall \bar{x})(\exists y)\varphi(\bar{x}, y) \qquad \Longrightarrow \qquad (\forall \bar{x})\varphi(\bar{x}, f(\bar{x})) \qquad f \text{ new}.$$

Then for any $\Theta \cup \{\psi\} \subseteq \mathrm{Sen}(\mathcal{L})$:

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Let K be an elementary class of $\mathcal{L}$-structures, $\mathcal{L}'$ an extension of $\mathcal{L}$, and K$'$ the class of $\mathcal{L}'$-structures whose $\mathcal{L}$-reducts are in K.

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Part VI

Eliminations and Applications

The Idea

Question. How can we prove that φ **preserves** Σ **in** K?

An Answer.

(a) Give a **proof system** that checks for a given $\psi \in \Sigma$ whether

$$\mathrm{Th}(\mathrm{K}) \cup \{\varphi\} \models \psi.$$

(b) Show that "applications of φ" can be **eliminated** from proofs.

Let us begin with some simple observations for **lattices**.

S. Negri and J. Von Plato. Proof systems for lattice theory.
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A Proof System GLat for Lattices

Axioms

$$\frac{}{s \leq s}\ (\text{ID})$$

Cut rule

$$\frac{s \leq u \quad u \leq t}{s \leq t}\ (\text{CUT})$$

Left rules

$$\frac{s_1 \leq t}{s_1 \wedge s_2 \leq t}\ (\wedge\Rightarrow)_1$$

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Theorem

(a) $\vdash_{\mathrm{GLat}} s \leq t$ *iff* $\models_{\mathrm{Lat}} s \leq t$.

(b) GLat *admits cut-elimination.*

A Proof System GLat for Lattices

Axioms

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Example: Boundedness in Lattices

Consider the following $\mathcal{L}_{\text{Lat}}$-sentence for expressing **boundedness**:

$$\varphi_{bd} = (\exists x)(\exists y)(\forall z)((x \leq z) \sqcap (z \leq y)).$$

Skolemizing this sentence gives

$$\text{sk}(\varphi_{bd}) = (\forall z)((\bot \leq z) \sqcap (z \leq \top))$$

in a language $\mathcal{L}^{b}_{\text{Lat}}$ containing extra constants $\bot$ and $\top$.

We consider GLat extended with the rules:

$$\frac{}{\bot \leq t}\,(\bot\Rightarrow) \qquad \text{and} \qquad \frac{}{s \leq \top}\,(\Rightarrow\top).$$

Theorem

(a) φ_{bd} *preserves the set of* $\mathcal{L}_{\text{Lat}}$*-equations in* Lat.

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in a language $\mathcal{L}^{u}_{\mathrm{Lat}}$ with extra unary function symbols $\downarrow$ and $\uparrow$.

We consider GLat extended with the rules:

$$\frac{u \le \downarrow u}{s \le t}\ (\le\downarrow) \qquad \text{and} \qquad \frac{\uparrow u \le u}{s \le t}\ (\uparrow\le).$$

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Density

Consider the variety G of **Gödel algebras** and the following $\mathcal{L}$-sentence φ expressing *linearity* and *density*:

$$(\forall x)(\forall y)(\exists z)(((x \leq y) \sqcup (y \leq x)) \sqcap (((x \leq z) \sqcup (z \leq y)) \Rightarrow (x \leq y))).$$

Skolemizing, we obtain the sentence

$$(\forall x)(\forall y)(((x \leq y) \sqcup (y \leq x)) \sqcap (((x \leq d(x,y)) \sqcup (d(x,y) \leq y)) \Rightarrow (x \leq y))).$$

in a language $\mathcal{L}^d$ containing an extra binary function symbol d.

Theorem

(a) *φ preserves the set of $\mathcal{L}$-equations in* G.

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A Proof System for Lattices

Axioms

$$\frac{}{t \Rightarrow t}\ (\text{ID})$$

Cut rule

$$\frac{s \Rightarrow u \quad u \Rightarrow t}{s \Rightarrow t}\ (\text{CUT})$$

Left rules

$$\frac{t_i \Rightarrow s}{t_1 \wedge t_2 \Rightarrow s}\ (\wedge\Rightarrow)_i \quad i = 1, 2$$

$$\frac{t_1 \Rightarrow s \quad t_2 \Rightarrow s}{t_1 \vee t_2 \Rightarrow s}\ (\vee\Rightarrow)$$

Right rules

$$\frac{s \Rightarrow t_1 \quad s \Rightarrow t_2}{s \Rightarrow t_1 \wedge t_2}\ (\Rightarrow\wedge)$$

$$\frac{s \Rightarrow t_i}{s \Rightarrow t_1 \vee t_2}\ (\Rightarrow\vee)_i \quad (\text{i} = 1,2)$$

A Sequent Calculus for Distributive Lattices

Axioms

$$\frac{}{\Gamma, t \Rightarrow t}\ (\text{ID}) \qquad \frac{}{\Gamma, \bot \Rightarrow t}\ (\bot\Rightarrow)$$

Cut rule

$$\frac{\Gamma_1 \Rightarrow u \quad \Gamma_2, u \Rightarrow t}{\Gamma_1, \Gamma_2 \Rightarrow t}\ (\text{CUT})$$

Left rules

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A Sequent Calculus for Heyting Algebras

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A Hypersequent Calculus for Heyting Algebras

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$$\frac{}{\mathcal{G} \mid \Gamma, t \Rightarrow t} \text{(ID)} \qquad \frac{}{\mathcal{G} \mid \Gamma, \bot \Rightarrow t} (\bot\Rightarrow)$$

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$$\frac{\mathcal{G} \mid \Gamma, t \Rightarrow s}{\mathcal{G} \mid \Gamma \Rightarrow t \to s} (\Rightarrow\to)$$

We obtain a hypersequent calculus GG for **Gödel algebras** by adding the **communication** rule:

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow s \quad \mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow t}{\mathcal{G} \mid \Gamma_1 \Rightarrow s \mid \Gamma_2 \Rightarrow t} \; (\textsc{com})$$

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The Density Rule

Let GG^{D} be GG extended with:

$$\frac{\mathcal{G} \mid \Gamma_1 \Rightarrow x \mid \Gamma_2, x \Rightarrow t}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow t} \ (\textsc{density})$$

where x does not occur in the conclusion.

Theorem

(a) $\vdash_{\mathrm{GG}^{\mathrm{D}}} \varphi \Rightarrow \psi$ *iff* $\varphi \leq \psi$ *in all dense linearly ordered Gödel algebras.*

(b) GG^{D} *admits density elimination.*

M. Baaz and R. Zach. Hypersequents and the proof theory of intuitionistic fuzzy logic. *Proceedings of CSL 2000.* LNCS 1862:187–201, 2000.

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What Can Go Wrong With Adding Density?

A calculus GCL for **classical logic** is obtained by extending GG with

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow t}{\mathcal{G} \mid \Gamma_1 \Rightarrow s \mid \Gamma_2 \Rightarrow t} \ (\textsc{split})$$

But then for *any* term t, we have a derivation in $\mathrm{GCL}^{\mathrm{D}}$:

$$\dfrac{\dfrac{\overline{x \Rightarrow x} \ (\textsc{id})}{\Rightarrow x \mid x \Rightarrow t} \ (\textsc{split})}{\Rightarrow t} \ (\textsc{density})$$

I.e., $\mathrm{GCL}^{\mathrm{D}}$ is *trivial* – as it should be.

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$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow t}{\mathcal{G} \mid \Gamma_1 \Rightarrow s \mid \Gamma_2 \Rightarrow t} \ (\text{SPLIT})$$

But then for *any* term t, we have a derivation in $\mathrm{GCL}^{\mathrm{D}}$:

$$\dfrac{\dfrac{\overline{x \Rightarrow x}\ (\text{ID})}{\Rightarrow x \mid x \Rightarrow t}\ (\text{SPLIT})}{\Rightarrow t}\ (\text{DENSITY})$$

I.e., $\mathrm{GCL}^{\mathrm{D}}$ is *trivial* – as it should be.

Concluding Remarks

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- However, there are interesting examples that fit better into a first-order framework.
- Establishing the admissibility of a rule can be useful.

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