Definition and regularity of quasiconformal mappings in metric spaces

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1 Introduction

These are notes of the graduate course I gave during the Fourth School on Analysis and Geometry of Metric Spaces held between 22-27 May in Trento, Italy. The interest for studying quasiconformal and quasisymmetric mappings in the general setting of metric spaces is motivated by the applications in obtaining rigidity results for quasi-isometries of hyperbolic spaces (see eg.in [BP02]). Since Bruce Kleiner gave a parallel set of lectures focusing on this aspect, I will not dwell too much on this point. The goal of my lectures is twofold: to present various regularity properties of quasisymmetric mappings on the one hand; on the other hand I will show that the infinitesimal notion of quasiconformality implies the global condition of quasisymmetry in quite general metric setting.

There are three type of regularity properties that are being considered: absolute continuity on curves, Gehring-type higher integrability results and Hausdorff dimension distortion by quasisymmetric maps. These subjects are being treated in the first part of the lectures. In this context I mention (without proofs) the sharp results of Kari Astala [Ast94] in the planar case. I will then try to present a more or less self contained proof of the Gehring's higher integrability result of quasisymmetric maps in general metric measure spaces with controlled geometry following the paper of Juha Heinonen and Pekka Koskela [HK98]. It is worth noting that beyond the planar case no sharp higher integrability result is known in Euclidean spaces. The second part of the lectures is based on the recent joint work with Pekka Koskela and Sari Rogovin. It is shown that the infinitesimal condition of quasiconformality implies the global property of quasisymmetry in Loewner spaces. In fact more is true: one can replace the usual definition of quasiconformality (based on a limsup condition) by an apriori much weaker liminf condition. (This was shown earlier by Juha Heinonen and Pekka Koskela [HK95] in the Euclidean setting and conjectured to hold in more general metric spaces.)

Acknowledgements I would like to thank the organizers: **Luigi Ambrosio, Bruno Franchi, Raul Serapioni** and **Francesco Serra Cassano** for the invitation. I thank also the lively audience in Trento for the attention, interest and challanging questions following my lectures. My gratitude goes to **Thomas Zürcher** for his hard work of writing up these notes and for clarifying many points of the often sketchy lectures.

2 QC Mappings. Definitions

There are several definitions for QC mappings. Let us start with the metric definition, which can be formulated in the general context of metric spaces.

Definition 2.1 (Metric definition for QC mappings). Let (X, d) and (X', d') metric spaces and $f\colon X \to X'$ be a homeo. between them. We define the upper and lower oscillation of f at the point x and scale $r > 0$ as:

$$L_f(x,r) := \sup\{d'(f(x), f(y)) : d(x,y) \le r\}$$
$$l_f(x,r) := \inf\{d'(f(x), f(y)) : d(x,y) \ge r\}$$

and the metric distortion quotient as:

$$H_f(x,r) := \frac{L_f(x,r)}{l_f(x,r)}.$$

Exercise 2.2. *a) Show that if X and X' are connected then for r small enough $H_f(x,r) \ge 1$.*

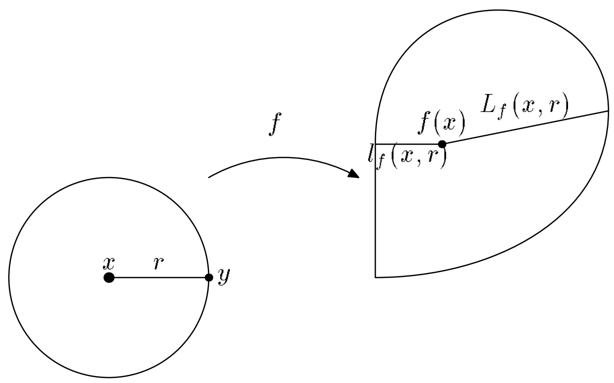
b) Find an example of spaces X, X' and a homeo. f for which $H_f(x,r) < 1$.

Definition 2.3 (metrically H-QC). We say that f is metrically H-quasiconformal (or, as we will say for short: metrically H-QC) if

$$\limsup_{r\to 0} H_f(x,r) \le H \text{ for all } x \in X.$$

We further let

$$l_f(x) := \liminf_{r\to 0} \frac{l_f(x,r)}{r}$$
$$L_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{r}.$$



Exercise 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ differentiable at $x \in \mathbb{R}^n$. We define the positive definite symmetric matrix $A(x)$ as*

$$A(x) = D^t f(x) \cdot Df(x). \tag{2.1}$$

Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $A(x)$, where

$$0 \le \lambda_1^2 \le \lambda_2^2 \le \cdots \le \lambda_n^2.$$

Show that $L_f(x) = \lambda_n$, $l_f(x) = \lambda_1$ *and*

$$H_f(x) = \left(\frac{\lambda_n}{\lambda_1}\right).$$

QC mappings are however typically non-smooth mappings. An important example in $\mathbb{R}^n$ is the so-called radial stretching defined in the following:

Exercise 2.5. *Let* $0 < \alpha < 1$, *and* $f_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ *defined by*

$$f_\alpha(x) = |x|^{-\alpha} \cdot x.$$

(See Figure 1)
Prove the following statements:

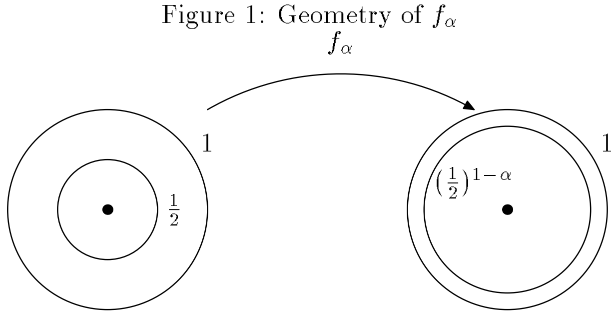
(1) $L_{f_\alpha}(x) = |x|^{-\alpha}$, $x \neq 0$

(2) $H_{f_\alpha}(x) = \frac{1}{1-\alpha}$ *if* $x \neq 0$ *and* $H_f(0) = 1$

(3) $f_\alpha \in W^{1,p}_{\mathrm{loc}} \iff p < \frac{n}{\alpha}$.

If we consider the image of the ring centered in 0 and defined by the two radii $\frac{1}{2}$ and 1, we get under the function f from Exercise 2.5 the ring with the same center but with radii $\left(\frac{1}{2}\right)^{1-\alpha}$ and 1.

Figure 1: Geometry of f_α



We can use the radial stretching f_α as building block to construct QC mappings which change dimensions of Cantor sets in a rather arbitrary fashion. Such construction was performed in the Euclidean setting by F. Gehring and J. Väisälä in 1973. We shall briefly sketch their construction in what follows.

Theorem 2.6 (F. W. Gehring, J. Väisälä 1973). *For any* t_1 *and* t_2 *with* $0 < t_1 < n$, $0 < t_2 < n$ *there are Cantor sets* C_1 *with* $\dim_H(C_1) = t_1$ *and* C_2 *with* $\dim_H(C_2) = t_2$ *and a QC map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $f(C_1) = C_2$.

Let B the unit ball and let us fix $N \in \mathbb{N}$ and $r > 0$ and assume that we have N disjoint balls $B(q_i, r) \subset B$. Associate to every ball B_i the conformal map f_i which is the composition of a translation and a scaling such that $f_i(B(q_i, r)) = B$.
Consider the iterated function system generated by $\{f_1^{-1}, \ldots, f_N^{-1}\}$. Fix t, $0 < t < n$. By choosing N and r in an appropriate way, we get that the dimension of the invariant set is t.
For given t_1, t_2, where $0 < t_1 < t_2 < n$ we construct via the system $\{f_1, \ldots, f_N\}$ described above a set S_{t_1} with dimension t_1. Similarly with a system $\{g_1, \ldots, g_N\}$ we can obtain a set S_{t_2} with dimension t_2. (We can construct S_{t_1} and S_{t_2} in such a way, that the cardinalities of the iterated function systems agree.)
The construction of the quasiconformal map f with the property that $f(S_{t_1}) = S_{t_2}$ is done inductively. At the end we need to look at the limit of the sequence of the maps we generated in each step.
As you can see in Figure 2 (here $N = 2$), in the k^{th} step we have N^k rings contained in N^k balls. Outside the largest ball the function is assumed to be the identity. One uses compositions of the functions $\{f_1, \ldots f_N\}$ to map each ring to the left initial ring shown in the top part of Figure 2. Then we map this ring with the function f_α to the right one. We compose the obtained map with corresponding compositions of functions in $\{g_1^{-1}, \ldots, g_N^{-1}\}$ such that we get maps like this:

$$g_{i_k}^{-1} \circ g_{i_{k-1}}^{-1} \circ \cdots \circ g_{i_1}^{-1} \circ f_\alpha \circ f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}.$$

This is the idea behind the following theorem which can be found in [GV73]. However F. Gehring and J. Väisälä did such a construction for cubes. The ring construction like we did it here is shown (in the Heisenberg group) in [Bal01].

Remark 2.7. As shown in [Bal01] Theorem 2.6 holds also in the Heisenberg group with a similar proof.

We observe that in the preceding theorem there is a dependance between H, n, t_1, t_2.

Question 2.8. *What is the relation between H, t_1 and t_2?*

We shall give a partial answer to this question in the next section. We shall see that we have a complete answer for the case of planar quasiconformal mappings. In higher dimensions (and more general metric spaces) we have only partial results.

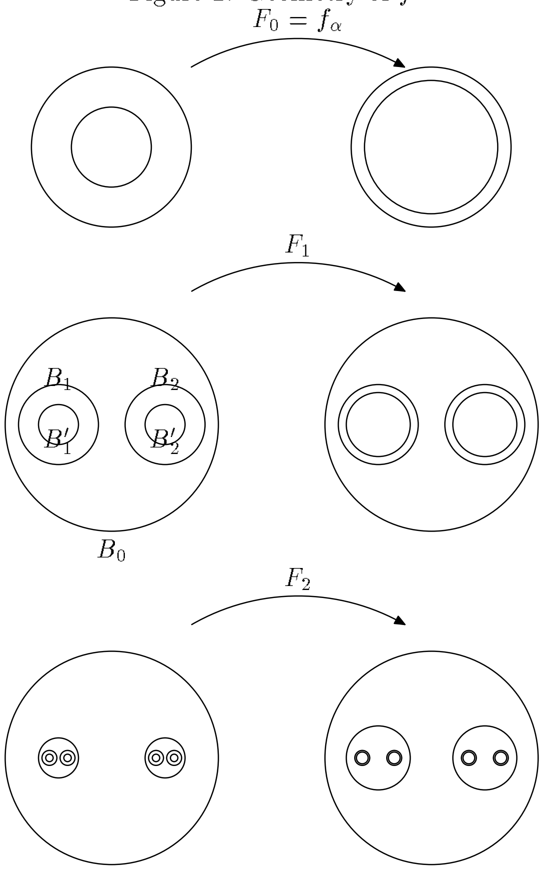
Question 2.9. *For which metric spaces do we have interesting QC maps?*

By interesting QC maps we mean here non-smooth or non-conformal mappings similar to those from Theorem 2.6.

3 QC Mappings in $\mathbb{R}^n$

In the previous section we considered QC maps in the general metric setting. In Euclidean space we have a lot more structure. In this section we recall some

Figure 2: Geometry of f



results on QC maps in Euclidean spaces. A good introduction provides the book of J. Väisälä, [Väi71]. First we would like to start with an alternative definition of QC maps: the analytic one. See for example [Ric93, Theorem 6.2, p.42].

Definition 3.1 (Analytic definition of quasiconformal mappings). Let $K \geq 1$. A homeo. f is K-QC if

(1) $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$ and

(2) $|Df(x)|^n \leq K J_f(x)$ for a.e. $x \in \mathbb{R}^n$,

where $J_f(x)$ stands for the determinant of $Df(x)$ which is the distributional Jacobian of f at x. In the sequel let us assume that K is the least K for which (2) holds.

Exercise 3.2. *This exercise is the continuation of exercises 2.4 and 2.5. Show that under the setting of Exercise 2.4 the following holds:*

$$K_f(x) = \frac{\lambda_n^{n-1}}{\lambda_1 \cdot \dots \cdot \lambda_{n-1}} \leq H_f^{n-1}(x).$$

Further calculate K_{f_α} for the function defined in Exercise 2.5

That the metric and the analytic definition of quasiconformality coincide in Euclidean spaces is shown by the following theorem.

Theorem 3.3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Then f is metrically QC if and only if f is analytically QC. Moreover*

$$K \leq H^{n-1}.$$

The analytic definition assumes some regularity for QC maps. In fact the regularity is higher than a priori expected. The following Theorem was proved by B. V. Bojarski for the case when $n = 2$ in 1955 in [Boj55]. In 1973 F. W. Gehring was able to prove a more general result, which can be found in [Geh73].

Theorem 3.4 (F. W. Gehring 1973). *Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-QC. Then there exists $p(K,n) > n$ such that f is in $W^{1,p}_{\text{loc}}$ for $p < p(K,n)$.*

Question 3.5. *What do we know about the dependance $(K,n) \mapsto p(K,n)$?*

Remark 3.6. Considering Exercises 2.4 and 2.5 we get by setting $f = f_\alpha$ that

$$p(K,n) \leq \frac{n \cdot K^{\frac{1}{n-1}}}{K^{\frac{1}{n-1}} - 1}.$$

F. Gehring conjectured that the upper bound for p in the above remark is really sharp:

Conjecture 3.7 (F. W. Gehring). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be K-QC . Then $f \in W^{1,p}_{\text{loc}}$ for*

$$p < p(K,n) = \frac{n \cdot K^{\frac{1}{n-1}}}{K^{\frac{1}{n-1}} - 1}.$$

Whether this conjecture is true or not is still an open question. However K. Astala proved the conjecture in the case where $n = 2$. The proof uses special techniques of complex analysis such as Beltrami equations, the Ahlfors-Beurling operator and holomorphic motions [Ast94].

Theorem 3.8 (K. Astala 1994). *If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is K-QC then $f \in W^{1,p}_{\mathrm{loc}}$ for $p < \frac{2K}{K-1}$.*

As a consequence of this theorem we obtain bounds for the Hausdorff dimension of the image of a compact set under a QC map answering Question 2.8 in the planar case.

Corollary 3.9. *Assume that Ω and Ω' are planar sets. Let $f\colon \Omega \to \Omega'$ a K-QC map and $E \subseteq \Omega$ compact. Then*

$$\dim_H f(E) \le \frac{2K \dim_H(E)}{2 + (K-1)\dim_H(E)}. \tag{3.1}$$

Moreover

$$\frac{1}{K}\left(\frac{1}{\dim_H E} - \frac{1}{2}\right) \le \frac{1}{\dim_H f(E)} - \frac{1}{2} \le K\left(\frac{1}{\dim_H E} - \frac{1}{2}\right).$$

In the same paper K. Astala showed the sharpness of the bounds by constructing sets such that inequality (3.1) becomes an equality:

Theorem 3.10. *For $0 < t < 2$ there exists a set $E_t \subset \mathbb{R}^2$ with $\dim_H(E_t) = t$ and a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is K-QC such that*

$$\dim_H f(E_t) = \frac{2K \dim_H(E_t)}{2 + (K-1)\dim_H(E_t)}.$$

After this overview of higher integrability and distortion of Hausdorff dimension in the planar case let us turn back to the result of Gehring.

The main idea of Gehring's proof of the higher integrability result in $\mathbb{R}^n$ is the following reverse Hölder inequality. To formulate Gehring's Lemma let us introduce first some notation. For a set $A \subset \mathbb{R}$ with $\mathcal{L}^n(A) > 0$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denote by

$$⨍_A g \, d\mathcal{L}^n = \frac{1}{\mathcal{L}^n(A)} \int_A g \, d\mathcal{L}^n.$$

Lemma 3.11 (F. W. Gehring 1973). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ a K-QC map. Then there exists a constant $b = b(K,n)$ such that*

$$⨍_{B(x,r)} L_f^n \, d\mathcal{L}^n \le b \left(⨍_{B(x,r)} L_f \, d\mathcal{L}^n \right)^n.$$

Remark 3.12. The sharp dependance from b of K and n is not known.

Now, that we know that the maximal derivative of a QC map satisfies a reverse Hölder inequality, the following lemma is the step we need.

Lemma 3.13 (F. W. Gehring, 1973). *Let $q > 1$, $\omega \in L^q_{\mathrm{loc}}$, $\omega \geq 0$. If there exists a constant $b \geq 1$ such that*

$$\fint_{B(x,r)} \omega^q \, d\mathcal{L}^n \leq b \left(\fint_{B(x,r)} \omega \, d\mathcal{L}^n \right)^q$$

then there is a $C > 0$ such that $\omega \in L^p_{\mathrm{loc}}$ for $q < p < q + C$, $C = C(q, b, n)$.

Gehring's Theorem 3.4 follows now as a combination of above lemmas by setting $\omega = L_f$ and $q = n$.

We shall see below that the method of Gehring also works in general metric measure spaces satisfying a Poincaré inequality for $1 \leq p < Q$.

4 QS Mappings. Definitions

The quasiconformality is a local condition. We get a global condition if we require $H_f(x, r)$ to be bounded instead of $\limsup_{r \to 0} H_f(x, r)$.

Definition 4.1 (Quasisymmetry). (1) We say that a homeo. $f\colon X \to X'$ between two metric spaces is quasisymmetric if there exists an $H \geq 1$ such that

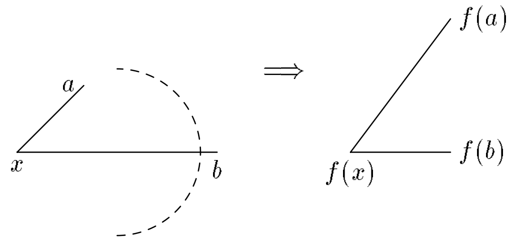
$$H_f(x, r) \leq H \quad \text{for all } x \in X,\ r < \operatorname{diam} X.$$

Observe that this can also be written as a three point condition. One requires

$$d(x, a) \leq d(x, b) \Longrightarrow d'(f(x), f(a)) \leq H d'(f(x), f(b)) \quad \text{for all } x, a, b \in X.$$

(2) A homeo. f as above is called η-quasisymmetric if there exists an increasing homeo. $\eta\colon [0, \infty) \to [0, \infty)$ such that

$$d(x, a) \leq t d(x, b) \Longrightarrow d'(f(x), f(a)) \leq \eta(t) d'(f(x), f(b)) \quad x, a, b \in X, t > 0.$$



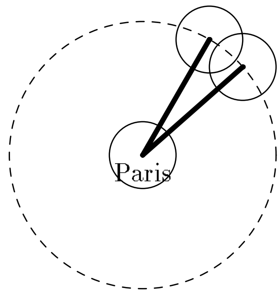
By setting $t = 1$ and $H := \eta(1)$ in the above definition we see that (1) is implied by (2). In spaces with good properties the converse is also true as shown by Jussi Väisälä in [Väi89].

Lemma 4.2 (J. Väisälä 1989). *Let X and X' two pathwise connected doubling (see Definition 4.3) spaces. Then f QS implies f η-QS.*

Definition 4.3 (Doubling Condition). A metric space (X, d) is called doubling if there exists a constant $M \in \mathbb{N}$ such that every ball $B(x, r)$ in X can be covered by M balls with radius $\frac{r}{2}$.

Example 4.4. (1) Most usual suspects are examples of doubling spaces: $\mathbb{R}^n$, Carnot groups, Riemannian manifolds, simplicial complexes, regular fractals.

(2) Infinite dimensional Hilbert spaces or the French railway are not doubling.



It is clear that the global condition of quasisymmetry is much more convenient to use than the a priori weaker condition of quasiconformality. The following question is therefore of crucial importance.

Question 4.5. *Under what conditions is quasisymmetry implied by quasiconformality?*

We postpone this important question for the last lectures. We assume at the moment that our mappings are quasisymmetric.

In the next section following the work of J. Heinonen and P. Koskela we shall introduce a class of spaces with good geometric properties where a reasonably good regularity theory of QS mappings can be obtained. In particular we shall present Gehring's higher integrability result for QS mappings in space with controlled geometry. (Theorem 5.6 below).

5 Higher integrability of QS mappings

Assume that f is QS and assume the following on (X, d, μ):

(1) The measure μ is Q-regular for a $Q > 1$. This means there exists a constant $C \geq 1$ such that

$$\frac{1}{C} r^Q \leq \mu\left(B\left(x, r\right)\right) \leq C r^Q.$$

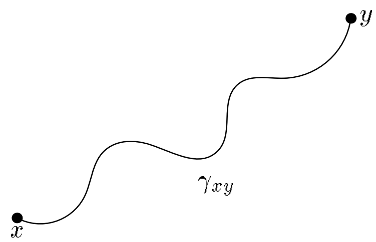
for every ball $B(x, r) \subset X$.

(2) (X, d, μ) satisfies a $(1,p)$-Poincaré inequality for some $1 \leq p < Q$, i.e. there are constants $C_1, C_2 \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq C_1 \operatorname{diam} B \left(\fint_{C_2 B} g^p \right)^{\frac{1}{p}},$$

where $u \colon X \to \mathbb{R}$ is continuous and $g \colon X \to [0, \infty]$ is an upper gradient of u. That means that for all points $x, y \in X$ and all rectifiable curves $\gamma_{x,y}$ connecting them, we have the following inequality:

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} g \, ds.$$



Remark 5.1. (1) By Hölder's inequality we see that we have the strongest Poincaré inequality if $p = 1$.

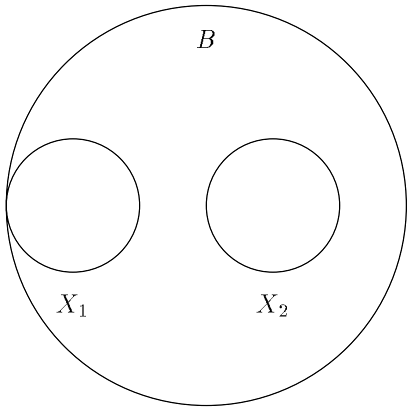
(2) Examples of spaces supporting a Poincaré inequality are again the usual suspects: the Euclidean spaces, Carnot groups, Riemannian and Sub-Riemannian manifolds.

(3) It is important to notice that the Poincaré inequality says much more than the doubling condition. It means in fact that the space is in some sense strongly connected.

Example 5.2. Let d the usual metric in the plain and $\mu = \mathcal{L}^2$ the 2-dimensional Lebesgue measure. Let further (X, d, μ) be the metric space with $X = X_1 \cup X_2 \subset \mathbb{R}^2$, where $X_1 = B((-4,0), 1)$ and $X_2 = B((4,0), 1)$. We consider the function u which is defined as the characteristic function of X_2. Then $u_B = \frac{\mu(X_2)}{\mu(X_1) + \mu(X_2)}$ for a ball B big enough to contain X_1 and X_2. Clearly $g \equiv 0$ is an upper gradient of u. We get

$$0 < \fint_B |u - u_B| \not< C_1 \operatorname{diam} B \left(\fint_{C_2 B} g^p \right)^{1/p} = 0$$

for all $C_1, C_2 > 0$. Therefore the space (X, d, μ) does not satisfy any Poincaré inequality.



The following exercise is a difficult one. It shows that even if the space has rather strong connectivity properties the Poincaré inequality may nevertheless fail.

Exercise 5.3. *Show that the Sierpinski carpet does not admit a $(1,p)$-Poincaré inequality for* *__any__* $p \geq 1$.

We would like to study derivatives of QS maps.

Definition 5.4 ($L_f(x)$, $\mu_f(x)$). If $f\colon (X,d,\mu) \to (X',d',\mu')$ is QS then we denote by

$$L_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{r}$$

the maximal derivative of f in x and by

$$\mu_f(x) := \limsup_{r\to 0} \frac{\mu'(f(B(x,r)))}{\mu(B(x,r))}$$

the so called volume or measure derivative.

Remark 5.5. Let $f_*\mu'$ the pull-back measure of μ':

$$f_*\mu'(E) := \mu'(f(E)), \quad E \subseteq X$$

then μ_f is the Radon-Nikodym derivative of $f_*\mu'$ with respect to μ:

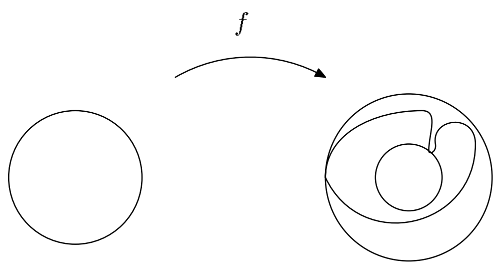
$$\mu_f = \frac{df_*\mu'}{d\mu}$$

and therefore $\mu_f \in L^1_{\text{loc}}$. So we know (see for example page 42 in [EG92])

$$\int_E \mu_f(x)\, d\mu(x) \leq \mu'(f(E)). \tag{5.1}$$

We have equality in (5.1) if and only if $df_*\mu' << d\mu$.
By Q-regularity and quasisymmetry we have

$$\left(\frac{L_f(x,r)}{r}\right)^Q \leq C\frac{\mu'(f(B(x,r)))}{\mu(B(x,r))}.$$



Letting $r \to 0$ we obtain

$$L_f^Q(x) \le C\mu_f(x) \text{ for } \mu\text{-a.e. } x \in X. \tag{5.2}$$

Therefore $L_f^Q \in L^1_{\text{loc}}$ or equivalently $L_f \in L^Q_{\text{loc}}$.

One of the most important results regarding regularity of quasisymmetric mappings in spaces with controlled geometry is the following theorem of J. Heinonen and P. Koskela in [HK98]. This result is a powerful generalization of Gehring's theorem.

Theorem 5.6. *Assume that*

- (X, d, μ) *and* (X', d', μ') *are locally compact metric measure spaces,*
- X, X' *are* Q*-regular for some* $Q > 1$,
- (X, d, μ) *admits a* $(1, p)$*-Poincaré inequality for some* $p < Q$ *and*
- $f : X \to X'$ *is* η*-QS.*

Then

- *there exists* $p_0 = p_0(X, X', \eta) > Q$ *such that*
- $L_f \in L^p_{\text{loc}}$ *for* $Q < p < p_0$,
- f *is absolutely continuous in measure:* $f_*\mu' << \mu$ *and*
- $\mu_f > 0$ *for* μ*-a.e.* $x \in X$.

Remark 5.7. The fact that $Q > 1$ in the preceding theorem has to be necessary. The Cantor function provides a counterexample in the case where $Q = 1$.

Proof of Theorem 5.6. The idea of the proof is to use a reverse Hölder inequality: we show that there exists $C > 0$ such that

$$\left(\fint_B L_f^Q \, d\mu\right)^{1/Q} \le C \left(\fint_B L_f^p \, d\mu\right)^{1/p}.$$

The problem is, that it is not clear if L_f is an upper gradient for $u(x) = d'(f(x), f(x_0))$, where x_0 is a fixed point in X. However if we define for $\varepsilon > 0$ the operator L_f^ε as

$$L_f^\varepsilon := \sup_{0<r<\varepsilon} \frac{L_f(x, r)}{r},$$

we get the following result:

Lemma 5.8. *Let $x_0 \in X$ be fixed. Then L_f^ε is an upper gradient for*

$$u(x) = d'(f(x), f(x_0)).$$

Proof. Let $\epsilon > 0$ be fixed and choose a rectifiable curve γ joining two points x and y in B. Suppose first that $d = \operatorname{diam}\gamma \le \varepsilon$. Then for each $z \in \gamma$ we have

$$L_f^\varepsilon(z) \ge \frac{L_f(z,d)}{d} \ge C^{-1}\frac{L_f(x,d)}{d}$$

by quasisymmetry. Thus

$$\begin{aligned}\int_\gamma L_f^\varepsilon\, ds &\ge C^{-1}\frac{L_f(x,d)}{d} l(\gamma) \ge C^{-1}L_f(x,d) \ge C^{-1}d'(f(x), f(y)) \\ &\ge C^{-1}\left|d'(f(x), f(x_0)) - d'(f(y), f(x_0))\right| = C^{-1}\left|u(x) - u(y)\right|.\end{aligned}$$

If $d = \operatorname{diam}\gamma > \varepsilon$, then pick successive points $x_0, \ldots, x_N$ from γ such that $x_0 = x$, $x_N = y$, and such that the diameter of γ_i, the portion of γ between x_{i-1} and x_i is less than ε for $i = 1, \ldots, N$. As above,

$$\begin{aligned}\int_\gamma L_f^\varepsilon\, ds &= \sum_{i=1}^N \int_{\gamma_i} L_f^\varepsilon\, ds \ge C^{-1}\sum_{i=1}^N d'(f(x_i), f(x_{i-1})) \\ &\ge C^{-1}d'(f(x), f(y)) \ge C^{-1}\left|u(x) - u(y)\right|.\end{aligned}$$

The lemma follows. □

We are also able to tell something about the integrability of L_f^ε:

Lemma 5.9. *L_f^ε belongs to weak-L^Q_{loc} with norm independent of ε i.e. for every ball B in X we have*

$$\mu\left\{x \in B : L_f^\varepsilon(x) > t\right\} \le Ct^{-Q}\mu'(f(B)).$$

Proof. Denote by E_t the set of points x in B where $L_f^\varepsilon(x) > t$. Then by Lemma 6.1, we can find a countable collection of disjoint balls $B_i = B(x_i, r_i)$ such that $0 < r_i \le \varepsilon$,

$$\frac{L_f(x_i, r_i)}{r_i} > t$$

and

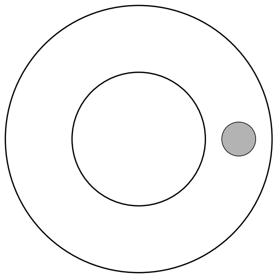
$$E_t \subset \cup 5B_i \subset 2B.$$

Thus by quasisymmetry and Q-regularity,

$$\begin{aligned}\mu(E_t) &\le C\sum r_i^Q \le Ct^{-Q}\sum L_f(x_i, r_i)^Q \\ &\le Ct^{-Q}\sum \mu'(f(B_i)) \le Ct^{-Q}\mu'(f(2B)) \le Ct^{-Q}\mu'(f(B)),\end{aligned}$$

as desired. Finally, because $L_f \le L_f^\varepsilon$, the lemma follows. □

Figure 3: Bound for ring



Let $B = B(x_0, R)$ and $u(x) = d'(f(x), f(x_0))$. By Lemma 5.8 and the Poincaré inequality it follows that

$$\fint_B |u - u_B| \, d\mu \le C_1 R \left(\fint_{C_2 B} \left(L_f^\varepsilon\right)^p \, d\mu \right)^{1/p}.$$

Letting $\varepsilon \to 0$ we see that L_f^p does in the Poincaré inequality the job an upper gradient normally does:

$$\fint_B |u - u_B| \, d\mu \le C_1 R \left(\fint_{C_2 B} L_f^p \, d\mu \right)^{1/p}. \tag{5.3}$$

Notice that by Lemma 5.9 and $p < Q$ we have that the right hand side of (5.3) is finite. If we only had a $(1, Q)$-Poincaré inequality we could not have concluded this, since L_f^ε is only in weak L^Q_{loc}. Next we would like to estimate the left hand side of the above inequality from below by an expression in L_f. By the quasisymmetry of f and the Q-regularity of μ we have:

$$u_B = \fint_B d'(f(x), f(x_0)) \, d\mu \ge \frac{1}{\mu(B)} \int_{B \setminus \frac{1}{2}B} d'(f(x), f(x_0)) \, d\mu \overset{QS}{\ge} \frac{1}{C} L_f(x_0, R).$$

For the last step note, that $\mu(B)$ and $\mu\left(B \setminus \frac{1}{2}B\right)$ are comparable by Q-regularity of μ. To see this (see Figure 3) note that the measure of the grey ball is comparable to the measure of the ring. Summarizing, we have obtained

$$u_B \ge \frac{1}{C} L_f(x_0, R). \tag{5.4}$$

For $0 < \delta < \frac{1}{C_2}$ small enough and $x \in \delta B$ by η-quasisymmetry of f the following estimate holds:

$$u(x) = d'(f(x), f(x_0)) \le \eta(\delta) L_f(x_0, R) \le \frac{1}{2C} L_f(x_0, R).$$

With (5.4) this gives

$$|u(x) - u_B| \ge \frac{1}{2C} L_f(x_0, R).$$

Now having estimated the integrand in (5.3), we turn to the integral on the left side of (5.3).

$$\int_{\frac{1}{C_2}B} |u - u_B| \, d\mu \geq \int_{\delta B} |u - u_B| \, d\mu \geq C^{-1} L_f(x_0, R) \, \mu(B).$$

This leads to an upper bound of $L_f(x_0, R)/R$. Using (5.3) we obtain

$$\frac{L_f(x_0, R)}{R} \leq \frac{C}{R} \fint_{\frac{1}{C_2}B} |u - u_B| \, d\mu \leq C \left(\fint_B L_f^p \, d\mu \right)^{1/p}$$

Now we are ready to tackle the reverse Hölder inequality:

$$\left(\fint_B L_f^Q \, d\mu \right)^{1/Q} \overset{(5.2)}{\leq} C \left(\fint_B \mu_f \, d\mu \right)^{1/Q} \overset{(5.1)}{\leq} C \left(\frac{\mu'(f(B))}{\mu(B)} \right)^{1/Q}$$
$$\leq C \frac{L_f(x_0, R)}{R} \leq C \left(\fint_B L_f^p \, d\mu \right)^{1/p}.$$

This is the reverse Hölder inequality we wanted to prove which implies the first statement of the theorem.
Let us prove the absolute continuity in measure. Starting from

$$\frac{L_f(x_0, R)}{R} \leq C \left(\fint_B L_f^p \, d\mu \right)^{1/p}$$

we obtain

$$L_f(x_0, R)^Q \leq C R^Q \left(\fint_B L_f^p \, d\mu \right)^{Q/p} \leq C R^Q \fint_B L_f^Q \, d\mu = C_B \int_B L_f^Q \, d\mu.$$

Since

$$(\operatorname{diam} f(B))^Q \leq C \int_B L_f^Q \, d\mu,$$

the absolute continuity in measure follows. By general property of weights satisfying a reverse Hölder inequality we obtain $\mu_f > 0$ a.e. □

In Theorem 5.6 the condition that $p < Q$ played an important role. It is a natural question to see what happens if $p = Q$. Results in this case are contained in J. Tyson's Ph. D. dissertation, see also [Tys98].

Theorem 5.10 (J. Tyson 1998-1999). *Assume*

- (X, d, μ), (X, d', μ') *are locally compact metric measure spaces, which are* Q*-regular for* $Q > 1$
- (X, d, μ) *admits a* $(1, Q)$*-Poincaré inequality*
- $f : X \to X'$ *is* η*-QS* .

Then

- (X', d', μ) *admits a* $(1, Q)$*-Poincaré inequality*
- f *is absolutely continuous in measure*

$$f_* \mu' << \mu.$$

6 Absolute continuity on curves

Besides higher integrability, an important regularity result for QS maps is the absolute continuity on curves.
A main technical device in what is to come are covering theorems. The following lemma is a classical covering theorem. See for example [Hei01] or [AT04].

Lemma 6.1 ($5r$-covering theorem). *Assume*

- (X, d) *locally compact and* $A \subset X$ *bounded*
- $\mathcal{F} = \{B = B(x, r)\}$ *is a family of balls in* X *such that* $A \subseteq \cup_{B \in \mathcal{F}} B$.

Then there's a finite or countable subfamily $\{B_i = B(x_i, r_i) : i = 1, 2 \ldots\}$ *of* $\mathcal{F}$ *such that*

- *the balls* B_i *are pairwise disjoint*
- $A \subset \cup_i 5B_i$, *where* $5B_i = B(x_i, 5r_i)$.

Remark 6.2. There are also similar results stating that A can be covered by balls with radii $3 + \varepsilon$ times the original ones. However, for our needs, the above statement suffices.
It's easy to see that one can also extract a subfamily such that the balls themselves cover A and the balls with radii one fifth of the original ones are disjoint.

We would like to prove absolute continuity of QS mappings on curves. Our statements will not hold for every curve but for almost every curve. For saying what almost every curve means, we must define an (outer) measure on curve families. The next definition provides us with such a measure.

Definition 6.3 (p-modulus of a curve family). Let (X, d, μ) be a metric measure space, $p \geq 1$ and $\Gamma \subset X$ a curve family. The p-modulus of Γ is defined as

$$\operatorname{mod}_p \Gamma := \inf_\rho \int_X \rho^p \, d\mu$$

where the infimum is taken over all admissible densities i.e. Borel functions $\rho \colon X \to [0, \infty]$ for which

$$\int_\gamma \rho \, ds \geq 1$$

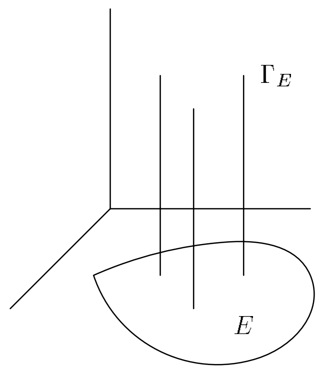
for all $\gamma \in \Gamma$.

Remark 6.4. Sometimes one considers in the above definition only rectifiable curves. Since the modulus of all unrectifiable curves is clearly zero, it does not matter if we restrict ourselves to rectifiable curves or not.

Exercise 6.5. *Let* $X = \mathbb{R}^n$*,* $E \subset \mathbb{R}^{n-1} \times \{0\}$*,* $\mathcal{L}^{n-1}(E) > 0$ *and define*

$$\Gamma_E := \{\{y\} \times [0,1] : \, y \in E\}.$$

Show that $\operatorname{mod}_p \Gamma_E > 0$ *for all* $p \geq 1$*.*



The result from [HK98] about absolutely continuity of QS mappings on curves is stated as follows:

Theorem 6.6 (J. Heinonen, P. Koskela 1998). *Assume that*

- (X, d, μ) *and* (X', d', μ') *are locally compact spaces which are* Q*-regular for some* $Q > 1$
- X *admits a* $(1,p)$*-Poincaré inequality for* $p < Q$
- f *is a QS homeo.* $f \colon X \to X'$

then f *is absolutely continuous on* Q*-a.e. curve in* X*, i.e. if*

$$\Gamma_0 = \{\gamma \colon I \to X : \, f \circ \gamma \colon I \to X' \text{ is not absolutely continuous}\}$$

then $\operatorname{mod}_Q \Gamma_0 = 0$*.*

Recall: $\gamma \colon I \to X$ is absolutely continuous if $E \subset I$ with $\mathcal{L}^1(E) = 0$ implies $\mathcal{H}^1(\gamma(E)) = 0$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure on X.

Proof of Theorem 6.6. Let $B \subset X$ be a ball. By Theorem 5.6 we know, that L_f is in $L^q(B)$ for some $q > Q$.

Claim 1. L_f^ε *is in weak* $L^q(B)$ *for* $L_f^\varepsilon(x) = \sup_{0<r\leq\varepsilon} \frac{L_f(x,r)}{r}$.

To show this, we define

$$A_t := \left\{x \in B : L_f^{\varepsilon}(x) > t\right\}.$$

Our goal is to show that

$$\mu(A_t) \le \frac{C}{t^q} \int L_f^q \, d\mu.$$

For every $x \in A_t$ we can choose a radius r_x such that

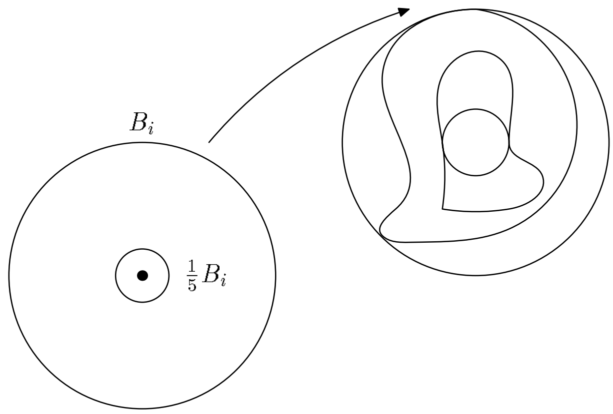
$$\frac{L_f(x, r_x)}{r_x} > \frac{t}{2},$$

or equivalently

$$\frac{1}{t^q}\left(\frac{L_f(x, r_x)}{r_x}\right)^q > \left(\frac{1}{2}\right)^q. \tag{6.1}$$

By Lemma 6.1 (respectively Remark 6.2) we can choose a subfamily of balls $B_i = B(x_i, r_i)$ such that

- $A_t \subset \cup B_i$
- $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$ for $i \neq j$.



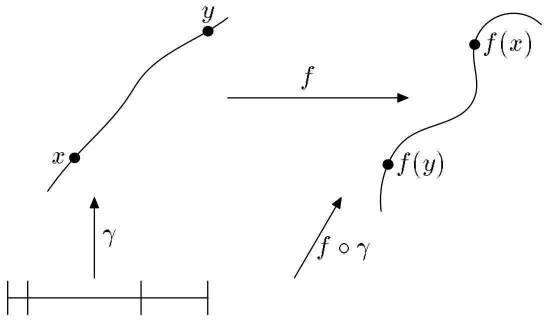
We obtain the following estimate (note that in inequality (6.1) the right hand side is just a constant and recall that μ is Q-regular).

$$\begin{aligned}\mu(A_t) &\le \frac{C}{t^q}\sum_i r_i^Q \frac{L_f^q(x_i, r_i)}{r_i^q} \le \frac{C}{t^q}\sum_i r_i^{Q-q}\mu\left(f\left(B\left(x_i, \frac{1}{5}r_i\right)\right)\right)^{q/Q} \\ &\le \frac{C}{t^q}\sum_i r_i^{Q-q}\left(\int_{\frac{1}{5}B_i} L_f^q\right)^{q/Q} \le \frac{C}{t^q}\sum_i \int_{\frac{1}{5}B_i} L_f^q \, d\mu \le \frac{C}{t^q}\int_B L_f^q \, d\mu.\end{aligned}$$

Claim 2. $\int_\gamma L_f^{\varepsilon} \, ds = \infty \quad \forall \gamma \in \Gamma_0$.

We assume by contradiction that $\int_\gamma L_f^\varepsilon \, ds < \infty$. Then since L_f^ε is an upper gradient of $d'(f(x), f(y))$ by Lemma 5.8, we have

$$d'(f(x), f(y)) \leq \int_{\gamma_{xy}} L_f^\varepsilon \, ds. \tag{6.2}$$



By inequality (6.2) and $\int_\gamma L_f^\varepsilon \, ds < \infty$ we conclude that $f \circ \gamma$ is absolutely continuous which contradicts that $\gamma \in \Gamma_0$.
For $\lambda > 0$ define $\rho_\lambda := \lambda \cdot L_f^\varepsilon$. By Claim 2 ρ_λ is admissible for Γ_0. This leads to a bound of the Q-modulus of Γ_0:

$$\int_X \rho_\lambda^Q \, d\mu = \lambda^Q \int_X \left(L_f^\varepsilon\right)^Q \, d\mu \to 0 \ (\lambda \to 0).$$

Consequently $\operatorname{mod}_Q \Gamma_0 = 0$ and thus the theorem is proved. □

Exercise 6.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be QS. Show that f is absolutely continuous on almost every line parallel to the coordinate axes.*
Hint: *Use Exercise 6.5 and Theorem 6.6.*

In the context of the Heisenberg group similar absolute continuity results were obtained by A. Korányi and H. M. Reimann in [KR95]. G. D. Mostow and G. A. Margulis generalized these results for Carnot-Carathéodory spaces, see [MM95].

7 Hausdorff dimension distortion

In this section we investigate how quasiconformal maps distort the Hausdorff dimension. The following Theorem can be found in [Bal01].

Theorem 7.1 (Z. B. 2000). *Assume that*

- (X, d, μ) *is a locally compact Q-regular space, $Q > 1$*
- $f\colon X \to X$ *QS and* $\mu_f \in L^p$, $p > 1$
- $A \subset X$ *a compact subset with* $\dim A = \alpha$.

Then the following upper bound for $\dim f(A)$ *holds:*

$$\dim f(A) \le \frac{Qp\alpha}{Q(p-1)+\alpha}.$$

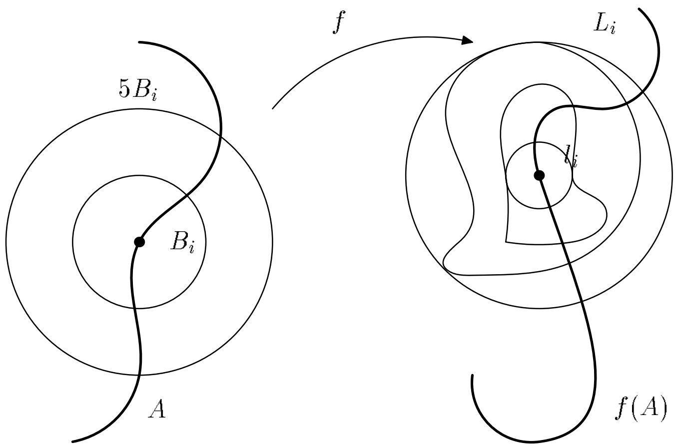
Proof. The case $\alpha = Q$ is clear. So assume that $\alpha < Q$. Choose $a > 0$ such that $\alpha < a < Q$. Thus we get

$$\frac{Q \cdot p \cdot \alpha}{Q(p-1)+\alpha} < \frac{Q \cdot p \cdot a}{Q(p-1)+a} =: b.$$

The statement follows from the following
Claim. $\mathcal{H}^b(f(A)) = 0$.
Since the dimension of A is α and $a > \alpha$ we get $\mathcal{H}^a(A) = 0$. For fixed $\varepsilon > 0$ and $d > 0$ we can find balls $B_j = B(x_j, r_j)$ such that

- $A \subset \cup_j 5B_j$
- $\operatorname{diam} f(5B_j) < d$
- $\sum_j r_j^a < \varepsilon$
- $B_i \cap B_j = \emptyset$, $i \neq j$.

Letting $L_i = L_f(x_i, 5r_i)$ and $l_i = l_f(x_i, r_i)$ we obtain by the quasisymmetry of f that $L_i \le Cl_i$ for a fixed constant C.



This gives us the following estimate

$$\operatorname{diam} f(5B_i) \le 2L_i \le C \cdot l_i \le C\mu(f(B_i))^{1/Q}.$$

Since by Theorem 5.6 f is absolutely continuous in measure, it follows by Hölder's inequality and the Q-regularity that

$$\mu(f(B_i)) = \int_{B_i} \mu_f \, d\mu \le Cr_i^{Q\frac{p-1}{p}} \left(\int_{B_i} \mu_f^p \, d\mu \right)^{\frac{1}{p}}.$$

This leads to

$$\sum_i (\operatorname{diam} f(B_i))^b \leq C \cdot \left(\sum_i r_i^a\right)^{\frac{b(p-1)}{a\cdot p}} \cdot \left(\sum_i \int_{\frac{1}{5}B_i} \mu_f^p \, d\mu\right)^{\frac{b}{p\cdot Q}} \leq C \cdot \varepsilon^t,$$

for a $t > 0$. Letting $\varepsilon \to 0$ conclude the claim. □

Remark 7.2. (1) Assume $X = \mathbb{R}^2$, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is K-QC. By Theorem 3.8 we know that

$$\mu_f = J_f \in L^p \quad \text{for } p < \frac{K}{K-1}.$$

We can calculate the following upper bound for the dimension:

$$\dim f(A) \leq \frac{2 \cdot \frac{K}{K-1} \cdot \dim(A)}{2 \cdot \frac{1}{K-1} + \dim(A)} = \frac{2K \cdot \dim(A)}{2 + (K-1)\dim(A)}.$$

(2) Since in general we don't know the precise exponent of integrability $p = p(n, K)$ it is useful to have another quantitative result describing the Hausdorff dimension distortion. In this direction we note the following sharp result on Hölder continuity of QC mappings. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-QC then $f \in \mathcal{C}^{\frac{1}{K^{n-1}}}$. This holds also in Carnot groups as shown in [BHT02].

Theorem 7.3 (I. Holopainen, J. Tyson, Z. B. 2002). *Let G be a Carnot group and $f\colon G \to G$ be K-QC. Then $f \in \mathcal{C}_{\text{loc}}^{\frac{1}{K^{Q-1}}}$.*

This theorem leads to the following corollary:

Corollary 7.4. *Under the assumptions of Theorem 7.3 we have*

$$K^{\frac{1}{1-Q}} \dim A \leq \dim f(A) \leq K^{\frac{1}{Q-1}} \dim A,$$

for $A \subset G$.

8 Quasiconformality and Quasisymmetry

We will later see that quasiconformality implies quasisymmetry in spaces which have nice connectivity properties. In this section we formulate these connectivity properties in terms of curve families in connection to the Poincaré inequality.

Definition 8.1 (Loewner space). Let (X, d, μ) be a Q-regular space. We say that X is Loewner if there exists a function $\Phi\colon (0, \infty) \to (0, \infty)$ such that

$$\operatorname{mod}_Q(E, F) \geq \Phi(t)$$

for all sets $E, F \subset X$ which are nondegenerate continua with

$$t \geq \Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}.$$

The expression $\Delta(E, F)$ is some sort of relative distance, and $\operatorname{mod}_Q(E, F) = \operatorname{mod}_Q \Gamma_{E,F}$, where $\Gamma_{E,F}$ is the curve family connecting E and F.

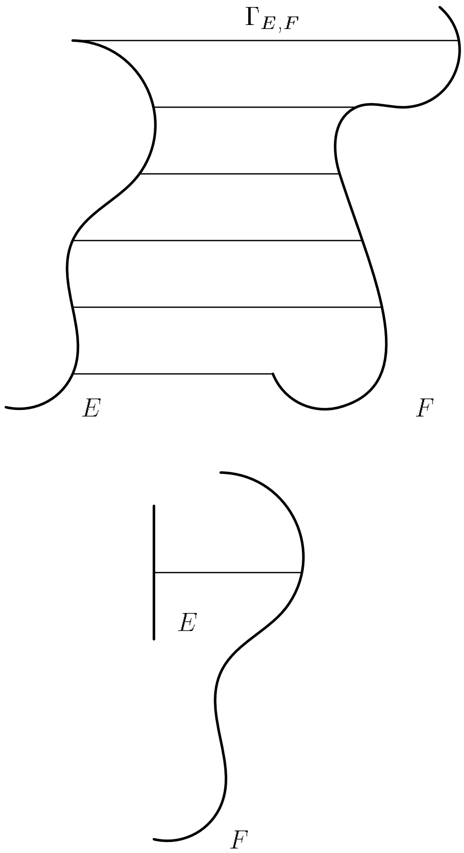


Figure 4: Space, which is Loewner

Remark 8.2. In Loewner spaces we know the asymptotic behavior of Φ.

$$\Phi(t) \approx \log \frac{1}{t} \quad t \text{ small}$$
$$\Phi(t) \approx (\log t)^{1-Q} \quad t \text{ large}.$$

Intuitively one can think that mod_Q is a measure for the conductivity in X.

As the following proposition shows, Loewner spaces have nice connectivity properties:

Proposition 8.3. *Assume that (X, d, μ) is Loewner. Then*

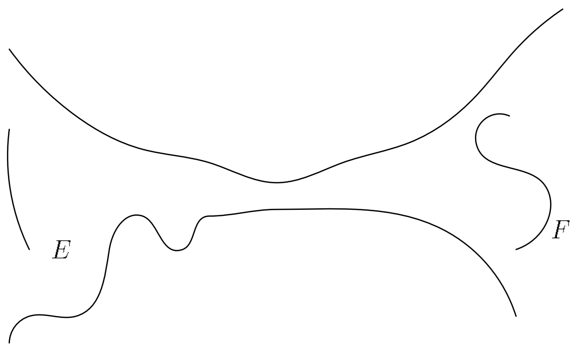


Figure 5: Space, which is not Loewner

- *X is LLC (locally linearly connected) i.e. there exists a constant $C \geq 1$ such that for all $x \in X$ and $r > 0$ any two points in $B(x, r)$ can be joined by a rectifiable curve in $B(x, Cr)$ and any two points in $X \setminus \overline{B}(x, r)$ can be joined by a rectifiable curve in $X \setminus \overline{B}(x, r/C)$.*
- *X is quasiconvex i.e. any two points in X can be joined by a curve whose length is no more than a fixed constant times the distance between the points.*

The following result of Heinonen and Koskela in [HK98] shows that there is an equivalence between Loewner spaces and spaces admitting a $(1, Q)$-Poincaré inequality.

Theorem 8.4. *Assume that (X, d, μ) is a locally compact, quasiconvex space which is Q-regular. Then it is Loewner if and only if it admits a $(1, Q)$-Poincaré inequality.*

We want now to investigate the connection between quasiconformality and quasisymmetry. Recall that $H_f(x, r)$ is defined as

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)}$$

and a mapping is QC if

$$H_f(x) = \limsup_{r \to 0} H_f(x, r) < H$$

for some $H > 0$. We would also like to consider the lim inf of $H_f(x, r)$.

Definition 8.5. Let f be a map between two metric spaces. Then we set

$$h_f(x) = \liminf_{r \to 0} H_f(x, r).$$

In Euclidean spaces, as J. Heinonen and P. Koskela showed in [HK95], the boundedness of h_f already implies quasisymmetry.

Theorem 8.6 (J. Heinonen, P. Koskela 1995). *Assume*

- $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a homeo.*
- $h_f(x) \leq H$ *for all* $x \in \mathbb{R}^n$.

Then f *is QS.*

The proof uses the Besicovitch covering theorem which enables us to select from a family of balls a nice subfamily with bounded overlap. However the Besicovitch theorem fails already in the Heisenberg group (see [KR95]). What can one prove in more general spaces than Euclidean ones? If one knows that the spaces are Loewner and H_f is bounded, for example the following theorem in [HK98].

Theorem 8.7 (J. Heinonen, P. Koskela). *Assume that*

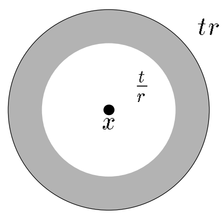
- X, X' *are Loewner spaces*
- $f : X \to X'$ *is a homeo.*
- $H_f(x) \leq H \quad \forall x \in X$, *for a* $H > 0$.

Then f *is locally QS.*

A natural question raised by Heinonen and Koskela in [HK95] is the validity of Theorem 8.6 in non-Euclidean settings such as of Carnot groups. A partial answer was provided in [BK00] using a quantity which is intermediate between h_f and H_f defined below.

Definition 8.8. Let f be a map between two metric spaces. We define for $t > 1$

$$H_f^t(x, r) = \sup_{\frac{1}{t}r < s < tr} \frac{L_f(x, s)}{l_f(x, s)}.$$



The result from [BK00] is stated as follows:

Theorem 8.9 (P. Koskela, Z. B. 2000).

- *Let* X *be* Q*-regular, locally compact, unbounded and Loewner*

- *Let $f\colon X\to X$ be a homeo. mapping bounded sets to bounded sets and assume that*
$$\liminf_{r\to 0} H_f^t(x,r)\le H \text{ for some } t>1.$$

Under these assumptions, f is QS.

The question of Heinonen and Koskela was finally settled in [BKR04] which we state as follows:

Theorem 8.10. *We assume*

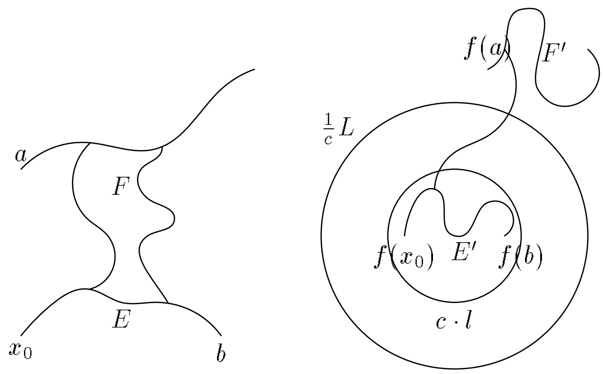
- *(X,d,μ) locally compact, unbounded and Q-regular for a $Q>1$*
- *(X',d',μ') locally linearly connected and unbounded*
- *X satisfies a $(1,Q)$-Poincaré inequality*
- *$f\colon X\to X'$ is a homeo. mapping bounded sets to bounded sets such that*
$$h_f(x)\le H \quad x\in X.$$

Then f is QS.

Remark 8.11. In the case when X satisfies a $(1,1)$-Poincaré inequality a stronger version of the result was proven in [BKR04] which allows a σ-finite $\mathcal{H}^{Q-1}$ measure exceptional set in the definition of quasiconformality.

Proof of Theorem 8.10. Our goal is to show the existence of a $H'\ge 1$ such that
$$d(x_0,a)\le d(x_0,b)\Longrightarrow d'(f(x_0),f(a))\le H'd'(f(x_0),f(b)).$$

Let $L:=d'(f(x_0),f(a))$ and $l:=d'(f(x_0),f(b))$. Assume $L>2c^2l$ (the other case is trivial), where c denotes the constant of the locally linear connectedness. Let E' be a continuum connecting $f(x_0)$ and $f(b)$ in $B(f(x_0),cl)$ and F' be a continuum connecting $f(a)$ and some point far away in $X\setminus B\left(f(x_0),\frac{1}{c}L\right)$. Set $E=f^{-1}(E')$ and $F=f^{-1}(F')$. By the Loewner property there exists a $C_0>0$ such that
$$C_0<\operatorname{mod}_Q(\Gamma_{E,F}). \tag{8.1}$$



We would also like to control $\operatorname{mod}_Q(\Gamma_{E,F})$ from above. The crucial result used in the proof is the following:

Lemma 8.12 (Main Lemma).

$$\operatorname{mod}_Q(\Gamma_{E,F}) \le C_1 \left(\log \frac{L}{l}\right)^{1-Q} \tag{8.2}$$

By (8.1) and (8.2) we obtain a constant C_2 such that

$$C_2 \le \left(\log \frac{L}{l}\right)^{1-Q},$$

which shows

$$\frac{L}{l} \le e^{C_2 \frac{1}{1-Q}},$$

and this is what we want to prove.

Proof of Main Lemma. Our task is to find $\rho\colon X \to [0,\infty]$ such that

(1) $\int_\gamma \rho\, ds \ge 1$ for all $\gamma \in \Gamma_{E,F}$

(2) $\int_X \rho^Q \, d\mu \le C_1 \left(\log \frac{L}{l}\right)^{1-Q}$.

But how can we find such a ρ? To indicate the idea of the proof we assume first that we are in the Euclidean setting: $X = X' = \mathbb{R}^n$ and $x_0 = f(x_0) = 0$. In this case the right admissible density ρ satisfying (1) and (2) is the following "logarithmic derivative" of f:

$$\rho(x) = \begin{cases} \left(\log \frac{L}{l}\right)^{-1} \cdot \frac{|Df(x)|}{|f(x)|} & x \in f^{-1}(A(L,l)) \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

where $A(L,l)$ is the ring domain $A(L,l) = \overline{B}(0,L) \setminus B(0,l)$. Let us check that ρ is an admissible function for the modulus. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a curve parameterized by arc length. Then

$$\begin{aligned} \left(\log \frac{L}{l}\right)^{-1} \int_\gamma \frac{|Df|}{|f|} \, ds &= \left(\log \frac{L}{l}\right)^{-1} \int_a^b \frac{|Df(\gamma(t))|}{|f(\gamma(t))|} \, dt \\ &\ge \left(\log \frac{L}{l}\right)^{-1} \int_a^b \frac{d \log |f(\gamma(t))|}{dt} \, dt \\ &= \left(\log \frac{L}{l}\right)^{-1} \log |f(\gamma(t))| \Big|_{t=a}^{t=b} \\ &= \left(\log \frac{L}{l}\right)^{-1} \log \frac{|f(b)|}{|f(a)|} \ge 1. \end{aligned}$$

Since $Q = n$ and $d\mu = dx$ is the Lebesgue measure:

$$\int_{\mathbb{R}^n} \rho^Q \, d\mu = \left(\log \frac{L}{l}\right)^{-Q} \int_{\mathbb{R}^n} \frac{|Df(x)|^Q}{|f(x)|^Q} \, dx.$$

Since f is QC we have $|Df(x)|^Q \leq H\mu_f(x)$ and therefore

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^Q \, d\mu &\leq H \left(\log \frac{L}{l}\right)^{-Q} \int_{\mathbb{R}^n} \frac{\mu_f(x)}{|f(x)|^Q} \, dx \\ &\leq H \left(\log \frac{L}{l}\right)^{-Q} \int_{A(l,L)} \frac{dy}{|y|^Q} \\ &= H \left(\log \frac{L}{l}\right)^{-Q} \log \frac{L}{l}, \end{aligned}$$

where we substituted $f(x)$ by y and assumed that $\rho = 0$ outside $f^{-1}(B(0, L))$.

Let's turn to the metric version of ρ. To discretize the expression (8.3) we have to use an appropriate covering $\mathcal{B}$ by "good balls" associated to the mapping f. Our formula for ρ then looks like:

$$\rho(x) = C \left(\log \frac{L}{l}\right)^{-1} \sum_{B \in \mathcal{B}} \frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}(f(x_0), f(B))} \cdot \chi_{2B}(x),$$

where $\mathcal{B}$ has to be a "good covering" by balls that will be defined below and C is an appropriately chosen large constant. Note that $\frac{\operatorname{diam} f(B)}{\operatorname{diam} B}$ plays the role of $|Df(x)|$ and $\frac{1}{\operatorname{dist}(f(x_0), f(B))}$ the one of $\frac{1}{|f(x)|}$.
What do we mean by "good covering"? Here are our required conditions on $\mathcal{B} = \{B(x_i, r_i)\}_{i \in I}$.

Definition 8.13.

(1) B_i should be "good balls" i.e. $f(B_i)$ should be roundish:

$$V_i := B\left(f(x_i), \frac{1}{2H} \operatorname{diam} f(B_i)\right) \subseteq f(B_i)$$

(2) $\cup_i f(B_i)$ covers $A(L, l)$

(3) $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset \quad i \neq j$

(4) $\frac{1}{10H} V_i \cap \frac{1}{10H} V_j = \emptyset$ for $i \neq j$.

To realize properties (1), (2) and (3) it is enough to apply the classical 5r covering lemma, [Hei01] and the condition $h_f(x) \leq H$. Property (4) however requires that the images of B_i are also separated in some sense, which is not guaranteed by the 5r covering theorem. To fix this problem we state a better covering lemma which takes care of the above difficulty.

Figure 6: property (1)

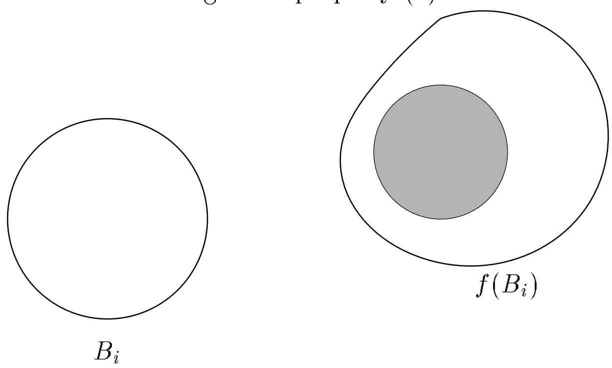
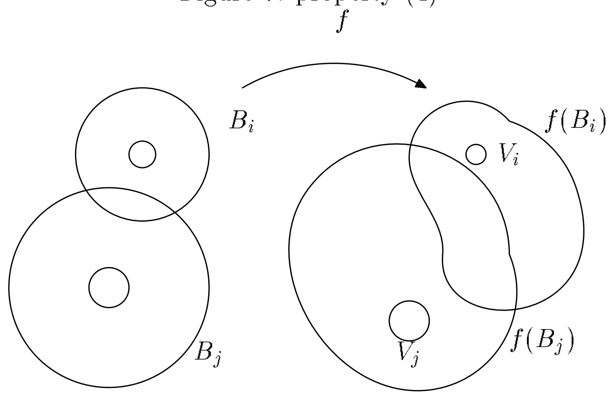


Figure 7: property (4)



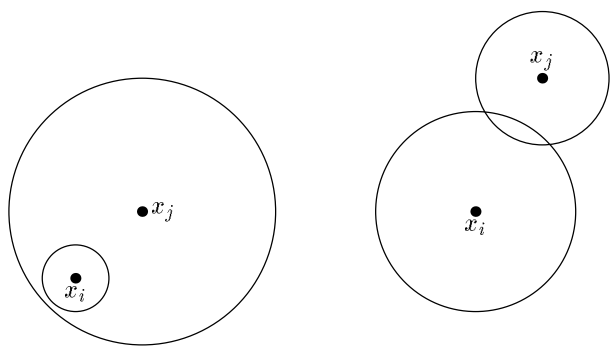
Lemma 8.14. *Assume*

- (X, d, μ) *is* Q*-regular*
- $A \subset X$ *bounded*
- *for* $x \in A$ *there's a ball* $B(x, r_x)$

Then there exists $\mathcal{B} = \{B_i : i = 1, 2, \ldots\}$*, finite or countable, where* $B_i = B(x_i, r_i)$ *such that*

1) $A \subseteq \cup_i B_i$
2) $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$, $i \neq j$
3) *for* (i, j), $i \neq j$ *one of the following two alternatives holds at least:*
   (a) $x_i \notin B(x_j, r_j)$ *and* $B(x_j, r_j) \setminus B(x_i, r_i) \neq \emptyset$
   (b) $x_j \notin B(x_i, r_i)$ *and* $B(x_i, r_i) \setminus B(x_j, r_j) \neq \emptyset$.

Figure 8: Example on the left does not fulfill condition 3). Example on the right does.



For the proof of Lemma 8.14 we refer to [BKR04].

We can use now the previous result to prove condition (4) for the covering $\mathcal{B}$ from Lemma 8.14.

Lemma 8.15. *We require*

- $f : X \to X'$ *a homeo. such that* $h_f(x) \leq H_f(x)$ *for* $x \in X$
- $\{B_i : i = 1, 2 \ldots\}$ *satisfies Lemma 8.14*
- *and*
$$V_i = B\left(f(x_i), \frac{1}{2H} \operatorname{diam} f(B_i)\right) \subseteq f(B_i).$$

In this case

$$\frac{1}{10H}V_i \cap \frac{1}{10H}V_j = \emptyset \quad i \neq j.$$

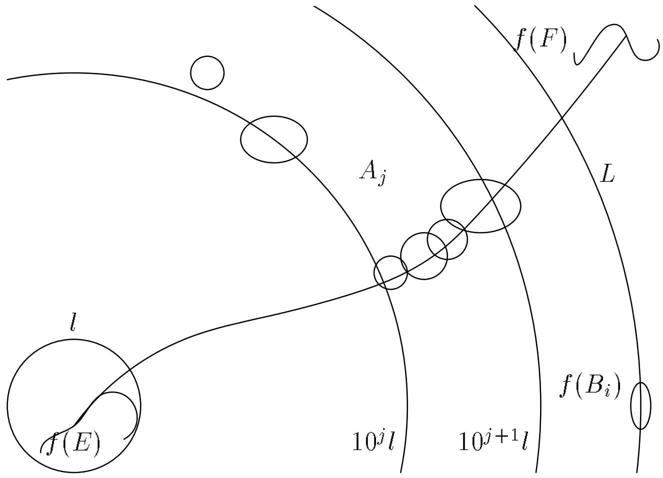
Suppose $\mathcal{B}$ is a good covering according to Definition 8.13 which is guaranteed by Lemma 8.15. Consider

$$\rho(x) = C\left(\log\frac{L}{l}\right)^{-1} \sum_{B\in\mathcal{B}} \frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}(f(x_0), f(B))} \cdot \chi_{2B}(x).$$

Suppose further that $\frac{L}{l}$ is huge and divide the interval between l and L:

$$l, 10l, 100l, \ldots, 10^{n_0}l \approx L,$$

where $n_0 \approx \log\frac{L}{l}$.



Choose the balls B_j so small that

$$\operatorname{diam} f(B_j) < \frac{1}{1000}l,$$

and divide $\mathcal{B}$ into parts as follows:

$$\mathcal{B}_j = \{B \in \mathcal{B} : f(B) \cap A_j \neq \emptyset\},$$

where $A_j = \overline{B}\left(f(x_0), 10^{j+1}l\right) \setminus B\left(f(x_0), 10^j l\right)$ is the ring with center $f(x_0)$ and radii $10^j l$ and $10^{j+1}l$.
We have to show:

1) $\int_\gamma \rho\, ds \geq 1$

2) $\int_X \rho^Q \, d\mu \le C \left(\log \frac{L}{l}\right)^{1-Q}$.

Condition 1) is shown by the following calculation:

$$\begin{aligned}
\int_\gamma \rho \, ds &= C\left(\log \frac{L}{l}\right)^{-1} \int_\gamma \sum_{B\in\mathcal{B}} \frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}\left(f(x_0), f(B)\right)} \chi_{2B} \, ds \\
&\ge C\left(\log \frac{L}{l}\right)^{-1} \sum_{B\cap\gamma\neq\emptyset} \frac{\operatorname{diam} f(B)}{\operatorname{dist}\left(f(x_0), f(B)\right)} \\
&\ge C\left(\log \frac{L}{l}\right)^{-1} \sum_{j=1}^{n_0} \sum_{\substack{B\cap\gamma\neq\emptyset \\ B\in\mathcal{B}_j}} \frac{\operatorname{diam} f(B)}{10^j l} \\
&= C\left(\log \frac{L}{l}\right)^{-1} \sum_{j=1}^{n_0} \frac{1}{10^j l} \sum_{\substack{B\cap\gamma\neq\emptyset \\ B\in\mathcal{B}_j}} \operatorname{diam} f(B).
\end{aligned}$$

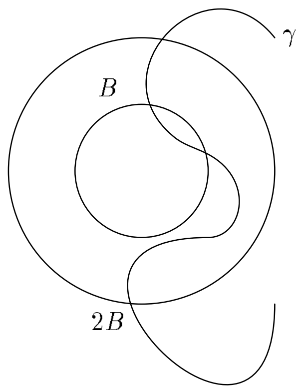
Noting that

$$\sum_{\substack{B\in\mathcal{B}_j \\ B\cap\gamma\neq\emptyset}} \operatorname{diam} f(B) \ge 10^j l,$$

we conclude

$$\int_\gamma \rho \, ds \ge C\left(\log \frac{L}{l}\right)^{-1} n_0 \ge 1$$

for C big enough.



The proof of 2) goes as follows:

$$
\begin{aligned}
\int_X \rho^Q \, d\mu &= C\left(\log \frac{L}{l}\right)^{-Q} \int_X \left(\sum_{B\in\mathcal{B}} \frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}(f(x_0), f(B))} \chi_{2B}\right)^Q d\mu \\
&\le C\left(\log \frac{L}{l}\right)^{-Q} \int_X \left(\sum_{B\in\mathcal{B}} \frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}(f(x_0), f(B))} \chi_{\frac{1}{5}B}\right)^Q d\mu \\
&= C\left(\log \frac{L}{l}\right)^{-Q} \sum_{B\in\mathcal{B}} \int_{\frac{1}{5}B} \left(\frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \cdot \frac{1}{\operatorname{dist}(f(x_0), f(B))}\right)^Q d\mu \\
&\le C\left(\log \frac{L}{l}\right)^{-Q} \sum_{B\in\mathcal{B}} \frac{(\operatorname{diam} f(B))^Q}{\operatorname{dist}(f(x_0), f(B))^Q} \\
&\le C\left(\log \frac{L}{l}\right)^{-Q} \sum_{B\in\mathcal{B}} \frac{\mu'(V_B)}{\operatorname{dist}(f(x_0), f(B))^Q} \\
&= C\left(\log \frac{L}{l}\right)^{-Q} \sum_{j=1}^{n_0} \sum_{B\in\mathcal{B}_j} \frac{\mu'(V_B)}{(10^j l)^Q} \\
&\le C\left(\log \frac{L}{l}\right)^{-Q} \sum_{j=1}^{n_0} \frac{(10^{j+1} l)^Q}{(10^j l)^Q} \le C\left(\log \frac{L}{l}\right)^{-Q} n_0 \\
&= C\left(\log \frac{L}{l}\right)^{-Q} \left(\log \frac{L}{l}\right) = C\left(\log \frac{L}{l}\right)^{1-Q}.
\end{aligned}
$$

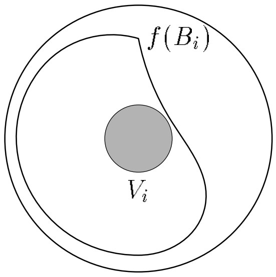
In the first inequality we used the following fact (see [Hei01, Exercise 2.10]):

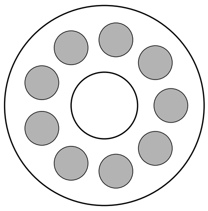
Exercise 8.16. *Suppose that $\mathcal{B} = \{B_1, B_2, \ldots\}$ is a countable collection of balls in a doubling space (X, d, μ) and that $a_i \ge 0$ are real numbers. Show that*

$$
\int_X \left(\sum_{\mathcal{B}} a_i \chi_{\lambda B_i}\right)^p d\mu \le C(\lambda, p, \mu) \int_X \left(\sum_{\mathcal{B}} a_i \chi_{B_i}\right)^p d\mu
$$

for $1 < p < \infty$ and $\lambda > 1$.
Hint: *Use the maximal function theorem together with the duality of $L^p(\mu)$ and $L^q(\mu)$ for $p^{-1} + q^{-1} = 1$.*

In the equality of the third row Condition (3) of $\mathcal{B}$ was used and next the regularity of μ has been applied. Afterwards we used the regularity of μ' and Condition (1) and (4).





□

□

If we permit an exceptional set E, we need some new ideas to prove the above result. The following Theorem is in [BKR04].

Theorem 8.17. *Let X and X' be proper, locally Ahlfors Q-regular metric spaces, $Q > 1$. Suppose that a homeomorphism $f\colon X \to X'$ satisfies $h_f(x) < \infty$ for all $x \in X \setminus E$, where E has σ-finite $(Q-1)$-dimensional Hausdorff measure and that $h_f(x) \le H < \infty$ almost everywhere. Then $f \in W^{1,1}_{\mathrm{loc}}(X;X')$.*

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